MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 96 No. 8

MEETING OF 1936 JUNE 10

Mr. J. H. Reynolds, President, in the Chair.

The President announced that His Majesty the King had been graciously pleased to sign the Roll of the Society as Patron.

The Treasurer made the following statement :—

"The Royal Astronomical Society is one of the twelve Learned Societies which are beneficiaries of the residuary estate of the late Herbert Spencer. The Society's share of the estate is expected to amount to at least £1000, of which £500 has already been paid over, on account.

"The conditions of acceptance of the legacy by the Societies, as stated in the Will, are that the sum received shall be spent in certain specified ways by the Governing Body within five years of the date of payment, and may not be used in any way for purposes of endowment.

"An epidiascope has already been purchased out of this bequest, also a carpet for the Council Room."

The election by the Council of the following Fellow was duly confirmed :—

Frank Mercer, Eywood Road, St. Albans, Herts.

Eighty-nine presents were announced as having been received since the last meeting, including :—

Washington, History of Science Society, Osiris, vol. i (David Eugene Smith Presentation Volume) : A Volume of Studies on the History of Mathematics and the History of Science.

Dr. August Kopff delivered the George Darwin Lecture, taking as his subject "Star Catalogues, especially those of Fundamental Character."

Professor Hertzsprung addressed the Meeting on the subject of the Leiden Observatory Eclipse Expedition.

STAR CATALOGUES, ESPECIALLY THOSE OF FUNDAMENTAL CHARACTER.

(*George Darwin Lecture, delivered by Dr. A. Kopff on* 1936 *June* 10.)

The George Darwin Lectures given in recent years have set before you various problems of modern astronomy, problems of spectroscopy or of the constitution of the universe—problems which are at the centre of our interest. What I am going to say may seem to be concerned with the classical astronomy of the past, and not with unsolved problems; but actually it again became, at the beginning of the present century, a problem of the first importance, which will demand all our efforts in the future.

You know that the problem of stellar dynamics is quite a new one, and we are only at the beginning of it. Poincaré compared the system of the stars with a gas in the stationary state; but Kapteyn for the first time was able to show—at the Meeting of the British Association for the Advancement of Science at the Cape in 1905—that the motions of the stars have their own peculiar laws, which can only be ascertained by observation.

So astronomers must observe the movements of the stars in all their details. But to study the movement of a body we must have a system of reference. Here astronomy meets a very great difficulty. To study the motion of the bodies of our solar system we use the stars as points of reference. Here our co-ordinates are practically given by the stars. Theoretically, you know, it should be possible to fix the system of reference by the planets alone in connection with Newton's law. But actually we use a mixed system of reference, which is obtained partly by the observation of the stars, partly by Newtonian mechanics. We use this same system also in studying the motion of the stars. If we had a complete knowledge of stellar dynamics, we could try to define a system of reference by the stars alone. In any case we must fix the system of reference for the stars by the same bodies whose motions we wish to determine.

In practice we use a limited number of stars as our system of reference in studying motion in the planetary system as well as in the stellar system. And this system of reference—the fundamental system or fundamental catalogue—is defined by the positions and proper motions of the stars contained in it together with the numerical value of the constant of precession. From these data we can calculate the co-ordinates of each fundamental star at any moment, and so can represent our fundamental system by these points of reference. The co-ordinates of any other point

at a given date can be obtained by a *relative* measurement. Since we are observing from the rotating Earth, we use the equator as a fundamental plane and the equinox as the zero point; both elements depend only on the Earth, its motion and the motion of its axis. As regards the equator and equinox, their position and motion are determined by observation in connection with Newton's mechanical law. But in determining the constant of precession we must further employ a hypothesis for the dynamics of the stars. So ultimately we have no ideal system of reference for the motion within the stellar system.

The different fundamental systems which exist now are the product of a historical development; and the way they were arrived at was doubtless the best that could have been adopted. Nearly all the observations were made by meridian instruments; only recently have some other methods been tried as an independent control for the system.*

Originally the only task was to obtain a system of reference for studying the movement of the bodies of the solar system; and both absolute and relative observations with meridian instruments gave the possibility of obtaining a network of star positions as points of reference over the whole sky. Often the observations were not made very systematically by the astronomers of the past, and many of these observations have no great value for the future. But the fact that we now possess good fundamental systems, and that we can now study the motion of the stars, we owe to the work of the astronomers of the past century. In our day, when we begin anew to turn our attention to the work of measuring star positions, we have a much clearer idea of what is necessary: we must obtain a fundamental system of reference free from systematic errors of any kind, and we must obtain the proper motions of stars well selected over the whole sky. We can now do the work more systematically; and we have organised it on a great scale, and must organise it in a yet more complete way in future.

As the product of the work of the nineteenth century we have different fundamental systems or fundamental catalogues. The best of these are the "Neue Fundamentalkatalog des Berliner Astronomischen Jahrbuchs" (N.F.K.) by Auwers (prepared by J. Peters, *Veröffentlichungen des Astronomischen Rechen-Instituts*, Nr. 33, 1910) and the Preliminary General Catalogue (P.G.C.) by Lewis Boss. Both catalogues have been revised, and the revision is completed. The catalogues of Newcomb and Eichelberger are of a more intermediate character.

Auwers' catalogue is the older one; it was built up in several steps. The first catalogue of a fundamental character was that published in 1879 for the northern sky as "Fundamental-Catalog für die Zonenbeobachtungen" (*Publikationen der Astronomischen Gesellschaft*, Bd. xiv). The places were obtained from several catalogues, especially those of Pulkowo and Greenwich, then just published; the system was that of the Pulkowo Catalogues of 1865. The proper motions were obtained by the comparison of the Greenwich observations of 1861 with those of Bradley in 1755. In the following years Auwers gave also a fundamental catalogue

* Especially by the Leiden Observatory (W. de Sitter, C. H. Hins and others).

for the southern sky ; he several times revised the results, and finally, in *Astronomische Nachrichten*, vol. **164** (1904) and vol. **168** (1905), he gave definitive corrections, and these data were used to form the N.F.K. The system was now a mean system formed by the instrumental systems of the absolute catalogues of the whole period of observation. Auwers gave a high weight to the oldest catalogues ; he partially re-reduced the old observations, for which he tried to discuss the instrumental errors as well as possible. Special attention was devoted to the new reduction of Bradley's catalogues. Auwers did this work during a period of twenty-five years, and his fundamental catalogue is the best result that could be obtained from the observations beginning with Bradley and Tobias Mayer and ending with the catalogues observed between 1890 and 1900.

Lewis Boss used nearly the same observational material as Auwers ; but he was able to profit by the work of Auwers, and he could reduce the observations at once and in a more homogeneous manner. He did not attach so much weight to the old catalogues as Auwers, and this seems to be an advantage in his work.

There is one great difference between the catalogues of Auwers and Boss. Auwers' catalogue contains 925 stars, that of Boss 6188 stars. Auwers took only stars with sufficiently good observations for positions and proper motions to define a fundamental system of high precision for the future. Properly we should compare Auwers' catalogue with the 627 standard stars of Boss, which alone define his system. Besides these, Boss included all stars for which he could then give proper motions of some exactness. The probable errors in Boss's P.G.C. give a good idea of the relative value of the data for the individual stars.

Both the catalogues of Auwers and of Boss define systems of reference which are equally good. Since both use Newcomb's constant of precession, these two catalogues are easily comparable for different epochs.

One word in regard to the constant of precession. What we observe are the positions of the stars relative to a moving equator and a moving equinox for different epochs. By comparing these positions we obtain the total motion of the stars, and we separate this motion into two parts : one part is the motion of the equator and equinox itself ; the other we call the proper motion of the star. If we change the constant of precession the proper motions are changed in consequence. We can separate the two motions only by postulating a law governing the motions of the stars. All determinations of the precessional constant in the past have postulated randomness of the stellar motions ; but even now, when we try to formulate more exactly the law of stellar motions, it does not seem necessary to change the numerical value of the constant. This was fully discussed a few years ago by K. Pilowski.* But naturally it may in future become necessary to make a change of this constant.

Perhaps there will be a way to decide this question in the not distant future. During the last few years a proposal of very great importance has been made, and already observers have begun to realise it. We have seen

* *A.N.*, 245, 121, 1932.

that we need, as points of reference for our fundamental system, bodies for which the law of motion is known. If we use very distant objects—stars or, much better, spiral nebulæ—we can assume that these bodies are practically at rest. As to the spirals, we find that they are objects outside our own stellar system ; they are regularly distributed in space, and we can suppose—as a first approximation—that the angular motion is in the mean practically zero. So the spirals afford a possible means of approximating to a fundamental system.

But we cannot build—at the present time at least—our fundamental system strictly on the spirals. We cannot make absolute observations of the spirals over the whole sky by meridian instruments ; and so we cannot obtain a homogeneous network of points of reference independent of the stars. At present we must always connect the spirals with the stars in their neighbourhood, and as a whole our fundamental system must be fixed by the stars. But the spirals as a system of bodies independent of the stars can be used to control and correct our fundamental system. So it is a very useful task for our observers to determine positions of the spirals in the system of a good fundamental catalogue. Perhaps the plates taken for the repetition of the A.G. Catalogues, and their continuation to the southern hemisphere, could be used to obtain for a number of brighter spirals positions simultaneous with those of the star positions in these catalogues. So we see a way in the near future to separate more exactly perhaps the apparent precessional motion from the real proper motion of the stars.

Since we cannot use in practice a system of reference formed by spiral nebulæ as a fundamental system, we must for the present represent our fundamental system by stars. But do our existing fundamental catalogues satisfy practical demands ? I think we must answer this question in the affirmative. I know that photography needs fainter stars as points of reference ; I know also that geodesy insists that the number of stars in our fundamental catalogues is not sufficient. But I consider that a fundamental catalogue cannot be expected to serve for such tasks of a specialised nature. For them we must find special ways to help ; but we will speak later upon this point in more detail.

The first purpose of a fundamental catalogue is to represent a fundamental system of reference for all astronomical problems, and this task can best be fulfilled by a fundamental catalogue of bright stars. I think the reasons for keeping up such a catalogue of bright stars, and not replacing them entirely by fainter stars, are especially these :—

(1) Our fundamental or absolute observations cannot be limited to night-time ; for various purposes we must observe also in twilight or day-time. Consider especially the observation of stars together with the Sun.

(2) For absolute observations smaller instruments are better than big ones, for the stability is much greater in the small instruments.

(3) To study the motion of the planets we must also observe them in daytime together with the stars.

(4) We must have most exact values of the proper motions of bright

stars, because we must study the motion of these individual members of our system in more detail.

(5) For exact time-service and other practical tasks we need bright stars.

So I think the historical way of arriving at a fundamental system, by forming fundamental catalogues of the kind we have now, was the right one ; and we should continue to go on in this way. And already both catalogues of Auwers and Boss have been developed in this way since their first appearance.

The development of Boss's catalogue goes on in a direct line. You know that the Albany General Catalogue, now finished by Benjamin Boss, will contain about 32,000 positions and proper motions in a revised fundamental system. The Albany General Catalogue thus has as its principal aim (as also had the P.G.C.) to give as many good proper motions as possible in a good system ; and so the new General Catalogue will be one of the most powerful aids in clearing up the dynamics of the stellar universe.

As I have already mentioned, Auwers gave in his fundamental catalogue only stars with well-defined proper motions. But he had not forgotten the other stars. His programme, however, was so extensive that it can be realised only in a long time, and there has always been a danger that it would not be fulfilled. He undertook the great work of the A.G. Catalogues. All stars down to the ninth magnitude were to be observed by meridian circles, first in the northern heavens and later also in the southern heavens. The observations were to be repeated half a century later. The observation of the primary A.G. Catalogues extended over a longer period than was expected ; and it seems very difficult—many astronomers even think it impossible—to use these observations for the future, although the F.C. of Auwers was especially prepared to obtain a homogeneous star system. But the great catalogue of Küstner (Bonn, 1900) is the first step in making these catalogues more homogeneous, and I think other steps must follow. The task is much less laborious than the *Carte du Ciel* (Astrographic Chart and Catalogue).

The other plan of Auwers, together with F. Ristenpart, was to collect all meridian observations of stars, beginning with Bradley and continuing up to 1900, to obtain all the existing data for studying stellar motions. It was not possible for them to compute the proper motions themselves.* We must be glad that the work now called the *Geschichte des Fixstern-Himmels* (G.F.H.) is finished for the northern sky, and that it will be continued by the Preussische Akademie der Wissenschaften at Berlin also for the southern sky.

This was the contribution of Auwers to our great problem of studying the stellar motions. Half a century had passed when the Astronomische Gesellschaft took up the plan of repeating the A.G. Catalogues of the northern hemisphere. This was consequently the right moment to revise the fundamental systems of Auwers ; new observations of about twenty-five to

* Some of the proper motions of the G.F.H. stars were calculated by the Observatory of Hamburg-Bergedorf.

thirty years were at hand. The latter work was begun at the end of 1924 by the Astronomisches Rechen-Institut of Berlin-Dahlem.

One question to be settled at the beginning of the work on the fundamental catalogue was: Should we prefer to build up an entirely new catalogue using all observations beginning with Bradley, or should we revise the existing one? If an entirely new catalogue was to be formed, it would be necessary to make a new system of standard stars combining all good catalogues of the past and, using this homogeneous system as a first approximation, improve it by all observations at hand; this new system could then be amplified by other stars. But really the whole of this work was already done by Auwers himself in a very complete manner. Indeed the N.F.K. is a fundamental catalogue of the highest quality, and we had to make only small corrections to it. We had also the advantage of using the manuscript computations of Auwers for all details; and so the only rational way was to revise Auwers' catalogue.

For this work I had the help of Dr. Kahrstedt and Dr. Heinemann, amongst others, in the discussion of systematic errors in both co-ordinates; for the other computations I had especially the help of the late C. Schoch and of Miss Nowacki. Allow me now to give some details about our work of revision and some experiences we made thereby. The result, the revised Auwers' fundamental catalogue, is called "Dritter Fundamental-Katalog des Berliner Astronomischen Jahrbuchs" (F.K. 3). The results are now partly published in the *Astronomische Nachrichten* * and in the Appendix to the *Berliner Jahrbuch* for 1936.

The work was done in two steps:

(1) The individual positions of the stars were corrected in the system of the N.F.K.
(2) The system of the N.F.K. was revised.

We first began with the individual correction of the star places without changing the fundamental system itself. We had to compare a series of modern catalogues, most of them published between 1900 and 1925, with the N.F.K. to obtain the difference between this system and the special system of each observed catalogue (called system of the instrument, or instrumental system). All catalogues which were observed in any way independently of an existing fundamental catalogue—which were in any sense absolute—could later be used to correct Auwers' system.

The observations themselves, brought into the system of N.F.K., gave the means of determining corrections for the individual stars. This part of the work was a very long one, and for most of the stars in the northern sky the corrections were very small. In the southern sky, and also rarely in the northern, there exist in our revised catalogue stars which are not sufficiently observed and need greater attention in the future.† In our

* For literature see *Vierteljahrschrift d. Astr. Ges.*, 68, 334, 1933. Also *A.N.*, 231, 345, 1928, and 242, 313, 1931.

† For example, star N.F.K. 310 = Br 1147 shows already great differences by different observers.

definitive publication of the F.K. 3 we shall give the mean errors of the individual corrections in the same form as these were given by Auwers.* Observers may examine these values and, by comparing with the data of Auwers, they can judge what increase of accuracy has been made in the individual star positions and proper motions.

Naturally the positions and proper motions of the individual stars have been distinctly influenced by the observations of the last 25 years. Yet for the N.F.K. of Auwers the positions were chiefly fixed by the observations between 1845 and 1895, whilst the proper motions depended more on the older observations beginning with Bradley. Now the modern series of observations beginning with 1845 has been prolonged for a quarter of a century ; and so the last 75 years of observations determine essentially the proper motions of the individual stars.

The same may also be said of the systematic corrections of Auwers' fundamental catalogue ; and this really means a great progress in our knowledge of the proper motions of the stars. I have already said that we formed the systematic differences between the N.F.K. of Auwers and the new observed catalogues. For systematic corrections we can use only catalogues of absolute character. For both co-ordinates, R.A. and Dec., these differences show the divergence of the N.F.K. from the modern instrumental systems very clearly, both when we go from the equator to the poles and when we go round through all hours of R.A. In the old discussions by Auwers as to the R.A., usually the mean of the differences between the F.C. and any observed catalogue was taken for all declinations from the equatorial zone to the pole. Now this mean has been taken in small successive zones of Dec. Very often these deviations have a different character near the equator and near the pole. The deviations between the N.F.K. and all the good modern catalogues of absolute character are found to be nearly the same. This means that the instrumental systems of these catalogues come—with some restrictions of which we shall speak later—very near to a true system ; and the differences found are to be accepted as real systematic corrections to the N.F.K. So the modern absolute catalogues on the average (in all there were about 25 in both hemispheres) provide in large measure the system of our revised fundamental catalogue. But it must be said that not all absolute catalogues make an essential contribution to these systematic corrections. The individual stars must be observed 6 to 8 times to eliminate the accidental errors of the observation for one night. Catalogues with only 2 or 3 observations are of much less value.

To correct systematically the proper motions it was necessary to compare the N.F.K. with the older catalogues of absolute character. Naturally these catalogues had already been used to obtain the proper motions of the N.F.K. But it was necessary to connect again these catalogues with those observed afterwards. In many cases we were compelled to compare once more the old catalogues with the N.F.K. Thereby it was always difficult to choose the catalogues which should be used to obtain systematic corrections to the proper motions. Generally we did not go back too far in the past.

* *A.N.*, 164, 225, 1904, and 168, 161, 1905.

Our procedure was different for different parts of the corrections ; but for the Dec. we used only absolute observations with meridian or vertical circles and not with any older instruments. In R.A. we could partly go back a little farther. The systematic corrections in proper motions are in the main determined by the absolute observations of about the last 75 years.

The oldest catalogues retain their value, especially for the individual proper motions of the stars in a given system. Systematically the old catalogues have no great value at the present time. Our discussion has shown that the systematic errors which were present in the positions of modern fundamental catalogues, such as the N.F.K., are the result of erroneous proper motions ; and the errors of the proper motions in these catalogues were produced by the systematic errors of the catalogues observed before, say, 1850.

Speaking now of the special systematic corrections in Dec. running from the equator to the poles,* we must first say that modern observed catalogues still show very serious systematic differences. The absolute instrumental systems of good catalogues have given for the position of the celestial equator (or of the equator point of each meridian plane, as we may say) values which differ by about 2″ from one to another. Differences of such character are not new. Both Auwers and Boss have found it, especially when comparing catalogues observed in the northern and the southern hemispheres. Both adjusted these differences ; Auwers mainly empirically, Boss using a formula whereby he supposed that the constant of refraction was erroneous at the different observatories. But if we try to use this last method for the modern catalogues, we obtain for the constant of refraction corrections which are quite impossible. There must be other sources of the discrepancies between the different catalogues. They must be sought in the instrument itself—what we generally call "Biegung" (flexure). The method of Bessel to eliminate the instrumental errors by changing the objective and ocular of the instrument is quite insufficient. W. Rabe at Munich † was able to show this immediately by comparing the method of Bessel with the old method of observing stars by direct and reflected light. We must therefore say that at present we cannot control fundamental instruments in such a way as to obtain a system of Dec. free from systematic errors.

The best way of finding the errors of an instrumental system in Dec. in present observations now seems to be the method of Newcomb, by using the observations of the Sun and the inner planets. The corrections thereby found for the different instrumental systems bring the equator point of all modern observed catalogues very near together ; in the mean the N.F.K. required in the neighbourhood of the equator a correction of only +0″·2. An independent investigation by H. R. Morgan at Washington ‡ confirmed definitely this value of the correction.

You know that various methods of obtaining a true system of Dec.

* See especially the papers of K. Heinemann, *A.N.*, 241, 145, 1931 ; 249, 289, 1933; 252, 353, 1934.

† *A.N.*, 248, 369, 1933.

‡ *A.J.*, 42, 149, 1933.

have been proposed by different astronomers for use in future. One group proposes to fix the equator point by observations of minor planets, both visually with the meridian circle and photographically (Dneprovsky, Numerov, Brouwer); the other group (Dneprovsky) will try to observe the same stars with the same instrument at two stations in the north and south hemispheres of the Earth. Both methods will certainly give a very useful contribution to our problem; especially they will give an independent means of correcting the instrumental system at a greater distance from the equator. But both methods cost much time and labour.

To obtain the course of the corrections of the proper motions of the N.F.K. in Dec. from the equator to the poles it was necessary, as we have already said, to compare anew the old catalogues (we began with Pulkowo 1845) with the N.F.K. It was possible to correct the equator points of these catalogues by the observations of the Sun. We possess observations of the Sun for Pulkowo beginning with 1845 and for Greenwich with 1860.

Near the equator * the differences between N.F.K. and fifty absolute catalogues from 1846 to 1926 have given a correction for the proper motions of $+0''.35$ per century. We were very gratified to check this value by the observations of the Sun made by Hornsby in Oxford between 1774 and 1798 which have been recently reduced by H. Knox-Shaw, J. Jackson and W. H. Robinson.† The difference between our F.K. 3 and Hornsby's observations is about $0''.05$, and is much smaller than for any other fundamental catalogue. For Dec. away from the equator ‡ we endeavoured to correct the instrumental system of each catalogue by always comparing one catalogue with groups of others at about the same epoch; for each catalogue the equator point had been corrected by observations of the Sun and planets, and it was assumed as a first approximation that the correction varied linearly from the equator value to a zero value at the pole. By then comparing one catalogue with a group of others a second approximation was obtained. I think there is now no other possible way of correcting the instrumental system of an old catalogue. The differences between the so-corrected absolute catalogues of the last 80 years and the N.F.K. gave the corrections which reduced the proper motions of the N.F.K. in Dec. systematically to the F.K. 3.

But it was also necessary to obtain additional systematic corrections for the Dec. and its proper motions depending on R.A. The corrections in Dec. cannot be constant through all hours of R.A., because the Dec. are observed during different seasons. Naturally our modern catalogues of absolute character are nearly free from seasonal influences; and so the comparison of N.F.K. with the modern catalogues, after applying the above-mentioned corrections, showed very distinct differences depending on R.A., which form a curve with two successive maxima and minima. These curves give immediately a correction of the system of N.F.K. in Dec. as good as we

* *A.N.*, **249**, 289, 1933.

† The Observations of the Rev. Thomas Hornsby, D.D., 1774 to 1798. Reduced by H. Knox-Shaw, J. Jackson and W. H. Robinson. Oxford, 1932.

‡ *A.N.*, **252**, 353, 1934.

can obtain at the moment. It is probable that our modern catalogues still contain some seasonal influence; and so it will partly remain in the F.K. 3.

In going back to the older * catalogues, beginning with Pulkowo 1845, to obtain the additional corrections for the proper motion in Dec., a very curious (or perhaps not curious) effect was to be seen. I give this as an illustration that the errors now found in the positions of the fundamental catalogue are produced by the errors of the proper motions. You may also see it clearly in other discussions.† The afore-mentioned curves, showing the differences between N.F.K. and the modern observed catalogues, have a maximum of amplitude at the present epoch; as we go back in time the amplitude becomes smaller, and between about 1860 and 1870 the differences pass through zero and then change their sign. This means that the modern catalogues, beginning with Pulkowo 1845, Cape 1850 and Greenwich 1860, have no great seasonal error in Dec. The errors in the N.F.K. of that character at the present time are due to errors in the proper motions, and arose through using observations which were obtained by older instruments. The older catalogues, especially the catalogue of Bradley 1755, must have had seasonal errors in Dec. which have influenced considerably our fundamental catalogues and their proper motions. It is remarkable that the catalogue of Bradley 1745 (later reduced by Auwers) is systematically better than the great catalogue of Bradley (1755).

Now, as regards the R.A., we must remark that it is much more difficult to correct a fundamental catalogue systematically in R.A. than in Dec. It is necessary to correct the equinox; to correct the R.A.'s in zones from the equator to the poles; to obtain additional corrections of seasonal character running through all hours of R.A.; and finally we must discuss the magnitude equation.

Much work on the correction of the equinox by observation of the Sun has been done in England, also at the Cape, at Pulkowo and at Washington. Dr. Kahrstedt ‡ has discussed anew the whole question, and it seems best to correct Newcomb's equinox by a constant amount $-0^s.05$. Dr. H. R. Morgan § has discussed the same problem independently and has obtained $-0^s.04$. The difference is, I think, determined by the circumstance that Dr. Kahrstedt finally used only observations with the impersonal micrometer, which seems to be more free from systematic errors, whereas Dr. Morgan used other observations also. The difference is not great. For F.K. 3 we have taken the value of Kahrstedt; Dr. Boss for the Albany G.C. has, if I understand right, taken the other value; there will therefore be a small constant difference between the two systems. Dr. Eichelberger in his catalogue added a correction for the centennial motion of Newcomb's equinox, so that the difference between his catalogue and the two other fundamental catalogues will become greater with time.

* *A.N.*, **245**, 49, 1932.

† Compare especially the two breaks in the N.F.K. discussed in *A.N.*, **252**, 378 and 385, 1934.

‡ *A.N.*, **244**, 33, 1931.

§ *A.J.*, **41**, 177, 1932.

To find the corrections of positions and proper motions in R.A.,* both in zones from the equator to the poles and through all hours of R.A., it is necessary to study in great detail the methods of observation used to obtain each catalogue. Most of our catalogues are not entirely absolute in R.A. Without regard to the equinox most catalogues use the clock stars of a given fundamental catalogue without any correction. If the observer has obtained the azimuth of the instrument free from the errors of the polar stars, such a catalogue, even if it is not independent of the clock stars, can be used to find by comparison with the positions of the fundamental catalogue its systematic correction in R.A. through all Dec. At the present time a number of modern catalogues exist well suitable for this purpose.

But to obtain the corresponding corrections of the proper motion, for which we are obliged to go back to old catalogues, we can really use only series of catalogues observed over a long period with the same unaltered instrument. If the azimuth is well determined, we can obtain the changes of the R.A. of the fundamental stars with sufficient accuracy only from catalogues observed with the same instrument—even if the instrumental system has small but constant errors. We possess only three series of catalogues of that sort: Greenwich (1860–1925), Pulkowo (1845–1915) and for the equator zone $-30°$ to $+40°$, also the Cape (1860–1905). We cannot say too emphatically that series of observations like these are a work of the highest value that astronomers can offer to their science.

Still more difficult is the situation as to the corrections in R.A. and the proper motion depending on R.A. Here also we need absolute observations independent of an existing fundamental catalogue; that is to say, we are not allowed to use the clock star places as they were adopted, but must correct them by the observations themselves. Two different methods were used to eliminate the systematic errors of the clock stars, which also form a curve with two successive maxima and minima. The one is the method used at Greenwich and the Cape of observing the clock stars in groups of many hours (12^h groups); the systematic errors in the system of the clock stars, forming our curve, are smoothed more and more if we adjust the R.A. over the whole time of observation. The second method is that of Pulkowo, used for the first time by Wagner in reducing the catalogue of 1865 and afterwards completed by other astronomers. Here the clock stars through all hours are divided in groups; the groups are connected with each other, and the observations of the different groups are corrected by adjustment. If we have a sufficiently good clock we soon obtain a good system of independent R.A.

The modern instruments and clocks give us absolute R.A. nearly free from systematic errors, so that we can correct the positions of the fundamental stars quite accurately enough by the observations of to-day. But to correct the corresponding values of the proper motion the material is very poor. We can only use old catalogues which are free from the periodic errors

* See the papers of A. Kahrstedt, *A.N.*, **228**, 57, 1926; **229**, 397, 1927; **232**, 89, 1928; **235**, 369, 1929; **237**, 17, 1929; **238**, 159, 1930; **240**, 385, 1930; **242**, 281, 1931.

of the clock stars. Near the equator we could only use the series of Pulkowo catalogues which cover a sufficient period of time. For higher Dec. we could also use the series of Greenwich, of the Cape, and partly also of Madras. The discussion of the errors of the proper motion in R.A. showed clearly that the systematic errors in the positions of the N.F.K. at the present day are produced by the errors of the proper motion, and these errors again come from the old catalogues observed before 1850. But the corrections which we could obtain for the proper motions are far from being definitive ; the observations at hand for this purpose are incomplete, and only the future can bring us nearer to an exact system.

Finally the N.F.K. of Auwers showed a magnitude equation * which could easily be corrected by a comparison with modern catalogues. It may perhaps be of interest that the magnitude equation of the N.F.K. has its greatest value between $-35°$ and $+40°$, becomes smaller between $+40°$ and $+60°$, and is practically zero north of $+60°$ and south of $-35°$.

I can give you only a brief report of the work that had been done to obtain the revised fundamental catalogue of Auwers, the F.K. 3. Systematically the F.K. 3 is better than the N.F.K. The positions, I think, which were corrected by a series of modern catalogues, can, with some exceptions, fully satisfy the usual demands ; the discussions of the observations made for the determination of time, for example, have already shown this.† The correction of the proper motion could be based on the series of modern observations beginning about 1850 ; and it is an improvement that the rejection of observations before that time can be compensated by the inclusion of modern catalogues between 1900 and 1925. But, as I have shown, the catalogues between 1850 and 1900 are not free from systematic errors ; and so it must be the task of the future to bring our knowledge of the proper motion more and more near to the truth. For in future the proper motion of the fundamental stars will still form the basis of the study of the stellar motions as a whole.

But what shall we do in future ? We need series of meridian observations of absolute character and of the highest accuracy possible. At the moment we can raise the accuracy only by studying the peculiarities of each instrument in all detail, by measuring all instrumental changes during the observations, by using only the best instruments (surely the crystal clocks will mean a conspicuous advance in the correction of our system in R.A.), and by studying yet more than ever the peculiarities of the observer or by eliminating him as much as possible. In recent years we have come much nearer to this requirement. For the catalogues of the nineteenth century it is sometimes impossible to say what system they have ; but now—especially for the meridian work of some pioneer astronomers—we know it full well.

As to the F.K. 3, we did not wish to stop at the revision of the Auwers stars. It seemed necessary to look more towards the future. Surely the selection of Auwers at his time was a very good one. He took all stars for which the proper motion could be obtained with highest accuracy, and the

* Papers of A. Kahrstedt, *A.N.*, **225**, 265, 1925 ; **227**, 49, 1926 ; **237**, 54, 1929.

† H. C. Freiesleben, *A.N.*, **258**, 265, 1936.

distribution was fairly regular through all hours of R.A. at the different Dec. But observers long ago found that the number of stars was insufficient, especially in the zones near the equator; and by Auwers' principle of selection the distribution was not regular with reference to the fundamental plane of our stellar system, the Milky Way. To remove these inconveniences and to insure the usefulness of Auwers' fundamental catalogue also for the future, a number (662) of additional stars were selected as "Zusatzsterne" and published in the *Berliner Jahrbuch* for 1936. Many of these are stars of other Almanacs; the rest were taken from Boss's P.G.C. or the Backlund-Hough list. Double stars with separation between 0″·5 and 5″—except Sirius, Procyon, Castor and α Centauri—were eliminated as improper for fundamental purposes. So in future we shall have a number of 1535 fundamental stars regularly distributed over the whole sky; this number is more than enough to secure our fundamental system.

One point perhaps must be specially mentioned. In selecting these additional stars it seemed necessary to look as much as possible to the future; that means that we selected stars, not because they were well observed in the past, but because they fill essential gaps in the system. A number of these stars can at the moment only be called fundamental stars with reserve; but if you look at the mean errors you will immediately distinguish them. The observations in the near future will soon correct these stars. It was really a great advance in Astronomy that at the meeting of the I.A.U. at Paris, 1935, the resolution was accepted that all Almanacs shall in future have the same fundamental catalogue, so that all stars will be observed in the same system; and we are gratified that this catalogue will be the revised fundamental catalogue of Auwers.

As I have already said, the Boss catalogue has taken a different development from that of Auwers. Its primary aim is to give as many stars as possible with sufficiently exact proper motions. The revision of Auwers' catalogue should alone correct the fundamental system. But the system of Boss's catalogue has also been revised and, as we can see by the corrections to P.G.C. published in the annual reports of the Dudley Observatory at Albany, the differences between the revised Boss and Auwers system will be very small.

But another work is proceeding in the same direction as that of the Albany G.C. As I have already said, it was part of the original plan of Auwers that the A.G. Catalogues should be repeated half a century afterwards. So the A.G. at its meeting at Potsdam in 1922 began to prepare this work for the northern sky, with the help especially of Küstner, Schorr and some other astronomers. The positions were to be obtained by photography, and the reference stars for measuring the plates were to be observed by different meridian circles. You know also that Dr. Schlesinger commenced the repetition of the A.G. Catalogues of the northern sky by photography in small zones. I do not think that it is useless to do this work twice. In the future astronomers will be glad to have had the northern sky measured twice. Especially we rejoice that now also the stars down to the ninth

magnitude of the whole southern sky are to be measured at the Cape Observatory and the station of Yale Observatory at Johannesburg; so that in the future we shall have a nearly homogeneous star catalogue of the whole sky down to the ninth magnitude.

But let me speak now only of the work done under the auspices of the A.G. which we call the "Second A.G. Catalogue" (A.G.K. 2). The meridian observations were made at Babelsberg and Bergedorf for the whole northern sky, and at Breslau, Pulkowo, Bonn, Leipzig and Heidelberg for smaller zones. Each star was observed twice at three different observatories. The photographs were taken at Bonn, Bergedorf and Pulkowo. The observations were obtained during about four years (1928–1932). The plates are measured, and the catalogue of the reference stars has been finished under the special direction of Professor Peters at the Astronomisches Rechen-Institut at Berlin-Dahlem.

The principal aim of the A.G.K. 2 was to obtain a homogeneous system of the places of the stars down to the ninth magnitude. Therefore the photographic plates have been taken nearly simultaneously with the observations of the reference stars; so that the proper motion of these are only required approximatively. Professor Küstner himself had selected about 13,750 reference stars of the seventh to ninth magnitude, also some fainter stars, quite regularly distributed over the northern sky—and these stars have been observed in the system of the revised Auwers' fundamental catalogue. So on every plate we have 15 to 20 stars of reference. You will find the details of the plan and of the work already done in the *Vierteljahrschrift* of the Astronomische Gesellschaft.*

One point I wish to emphasise in regard to the meridian work. To obtain the reference stars really in the system of the fundamental stars Küstner's series of fundamental stars were to be observed during the whole time of the meridian work. You know by experience that, even if the observer uses a fair number of fundamental stars as points of reference on each night, he always obtains the measured stars in a system which differs more or less from the fundamental system. What he obtains is in any case a system of his instrument. But now, if—as Küstner did first when observing his great catalogue of Bonn 1900—from time to time during the whole period of observations only fundamental stars are observed during the whole night (what we call Küstner's series), we can study the differences between the system of the instrument and the fundamental system itself in all details, and the observed stars can really be reduced to the fundamental system. Naturally the fundamental stars must extend in Dec. over the limits of the usually observed zones.

At the Rechen-Institut we have had in recent years the opportunity to compare the positions of reference stars observed in the old manner with those observed by using such series. The Eros comparison stars of first order were observed at different observatories in the regular way. If you look at the curves published in the *Astronomische Nachrichten* † you will see that

* Plan in *V.J.S.*, 62, 242, 1927. Also reports in *V.J.S.*, 61, 297, 1926; 63, 375, 1928; 65, 305, 1930; 68, 413, 1933; 70, 377, 1935. † *A.N.*, 241, 345, 1931.

they show systematically very great differences. (Only Dr. Dick, Babelsberg, had used Küstner's series.) The reference stars of A.G.K. 2 for which all observatories had used Küstner's series are practically in the system of the fundamental catalogue. So the system of the A.G.K. 2 is really as homogeneous as can now be attained.

In the future only two kinds of meridian observations should be made :

(1) Absolute observations free as much as possible from the positions of a fundamental catalogue.
(2) Relative observations referred to a given fundamental system, observed only in conjunction with Küstner's series.

In both cases it must clearly be stated what is the equinox and what the epoch of the catalogue, and also a comparison between the observed catalogue and a fundamental catalogue should be given by the observer (proposal by H. R. Morgan).

As to the reduction of the photographic plates, I will only say that Professor Peters * has studied in detail the possibility of using Hollerith calculation machines, and that it seems they will much reduce the time and labour spent on this work.

You see that at the moment we have a fundamental system of sufficiently good quality, which can be corrected from time to time. The Albany G.C. contains the proper motion of very many stars for the study of the motions of the Milky Way system. The G.F.H. gives us the possibility of obtaining the proper motions of many more stars if they are needed. Also Kapteyn's plan provides us with the proper motions of many faint stars. Soon we shall obtain new observations of stars down to the ninth magnitude in both hemispheres. And we must not forget the *Carte du Ciel*, which will provide much material for the study of stellar motions, if we are able to make the material homogeneous.

But what of the future ? We want our work on star catalogues to be more systematic than it has been in the past. I shall not speak of special needs for geodetic and other purposes ; here astronomy must help by special contributions. If we have in mind stellar dynamics, we must have the proper motion of stars of all magnitudes from the brightest to the faintest ones given in an exact system of reference.

As to the observations, we must limit the different tasks. The fundamental stars and, I think, also other bright stars must be observed with meridian instruments and other instruments which give absolute positions. The faint stars are to be observed photographically. Down to the ninth magnitude all existing stars can be observed in future ; for fainter stars a selection will be necessary. Also the spiral nebulæ should be observed regularly in the system of our fundamental catalogue.

But we must have a connection between meridian observations and photography by forming a catalogue of reference stars for the photographic plates. For the repetition of the A.G. Catalogue of the northern sky we

* *V.J.S. der Astr. Ges.*, **70**, 314, 1935.

had the selection of 13,750 stars made by Küstner. Dr. Schlesinger has, I think, for the first time proposed that the Albany G.C. with 32,000 stars over the whole sky should be in the future the standard catalogue of reference stars, and that only these stars should be observed by meridian instruments in future. As the distribution of the stars in the Albany G.C. is not sufficiently regular, the catalogue must be supplemented. A Sub-Committee of the Meridian Commission of the I.A.U. will do this work ; and it is to be hoped that it will be done soon after the appearance of the G.C. There is only one question which must be settled : Are the stars in the G.C. faint enough for photographic work ? We can, of course, use screens for the measurement of the brighter stars ; but we would prefer to have plates without screens. Küstner has selected—as already said—standard stars for A.G.K. 2 between 7^m and 9^m. For the selection of those stars we should look to the future—what we really need, and not what we have partly by chance. Certainly it must be our aim to have for the photographic work of various kinds in future a standard catalogue of about 30,000 stars regularly distributed in both hemispheres. As many stars as possible should be taken from the Backlund-Hough list and from Albany G.C., and other stars must be added from other sources.

But there is already another question. Will it be possible to observe these stars regularly with meridian instruments so that we obtain in a short time proper motions of high exactitude, so exact that these 30,000 stars can be used at every epoch as fundamental stars ? I do not believe this. And it seems for the present necessary to select from this standard catalogue about 3000 or probably more stars, forming a fundamental catalogue of faint stars between the sixth and ninth magnitude. We do not need current ephemerides of these stars ; but they must be observed, preferably in the near future, precisely in the system of our fundamental catalogue, free from systematic errors, to obtain as soon as possible good proper motions. I think that the value of such a catalogue does not lie only in the future, as providing a system of reference for the standard catalogue of the 30,000 stars. It lies much more in the past.

We need such a smaller catalogue of faint stars to exhaust soon the meridian observations of the nineteenth century. The systematic errors of the catalogues of this period are only determined for the bright stars, and are only accurate for these. At that time observers did not use screens or other methods of eliminating systematic errors ; they did not even know of these errors. But we can obtain good values of the differences between the catalogues of the past and a fundamental system only if the fundamental catalogue has stars of the same magnitude as the observed catalogue. So this fundamental catalogue of faint stars will be necessary in order to make the meridian work of the nineteenth century fruitful in furnishing exact proper motions for all material contained in the G.F.H.

We are now at the end. Star catalogues and their systematic relations are not very interesting for most astronomers, but I think they are fundamental for astronomical work. The proper motions form the chief data for our study of stellar dynamics ; they give us also the secular parallaxes which are

of the highest value for statistical purposes. I cannot speak of the third component of stellar movement, the radial velocity ; proper motion and radial velocity give us the absolute velocity in space. Very often, however, it is much better to discuss proper motion and radial velocity separately, because then the systematic errors of both components, obtained by widely differing methods, can be much better discussed.

As to stellar dynamics, we know we are only at the beginning, and we can only go from one step to the next. When studying the constitution of our universe, astronomers began with a very simple model ; stars were considered as being distributed uniformly in space. The contradiction with the observational results led finally to the model of Seeliger and Kapteyn ; and just at the moment when Charlier said that we should now develop the details of this model, Shapley was able to show that our then known universe was only a very small part of a model of much greater dimensions.

It seems we must proceed the same way as regards stellar motions. First we contemplated stars like atoms of a gas in a steady state ; then we found that the velocity distribution had its own peculiar law, which we hoped to explain by the mass distribution in the system of Seeliger and Kapteyn. Shapley's model brought us to the rotational theory of Oort and Lindblad, and we tried to understand the details of observed motion from this theory. Just lately astronomers have shown that any stream motion can give, in the vicinity of the Sun, the same effects as a rotation of the whole greater system. What shall we do? We must continue to study the motions in the universe, both near and distant, more exactly and more in detail ; and we are glad that we can go back to our star catalogues—often a little despised—as the foundation of our science.

A SELF-RECORDING MICROPHOTOMETER OF NEW DESIGN.

C. S. Beals.

In a previous paper the writer * described a direct reading photo-electric microphotometer which has since been in use at this observatory for some years and has given highly satisfactory service. In view of the nature of the work to be undertaken during the next few years, however, the decision was made some time ago to convert the instrument into one of self-recording type. The work of rebuilding was completed nearly a year ago, and as the general design of the new instrument differs radically from that of previous types it has seemed worth while to publish a brief description. The recording device was designed by Mr. S. S. Girling in collaboration with the writer, while the instrument was built by Mr. Girling in his machine shop in Victoria.

* *M.N.*, **92**, 196, 1932.

The Optical System.—The source of light is a six-volt bulb containing a single spiral filament, fed by two 90-ampere hour storage batteries. By means of a condenser mounted on the lamp housing an image of the filament is projected on to an adjustable slit, subsequently referred to as the first slit, situated 18 cm. from the condenser. An image of the illuminated slit is then formed on the photographic plate by an achromatic microscope objective of 30 mm. focus and 7 mm. aperture, mounted integrally with the first slit in a tube 14 mm. in length. A second achromatic lens of 16 mm. focus and 11 mm. aperture takes the diverging beam from the image on the plate and forms a second image on a white screen of area 20 × 20 cm. and 32 cm. distant from the plate. In the centre of this screen is the second slit, which opens symmetrically and is variable both in length and width. The image is brought into coincidence with the second slit by means of a horizontal motion of the unit containing the first slit and microscope objective. Behind the second slit is placed the light sensitive element.

For convenience in alignment of the plate and for the visual examination and measurement of spectra an auxiliary source of illumination is provided by a 100-watt 110-volt lamp mounted 30 cm. above the optical axis. A patch of light from this lamp is focussed on the plate by means of a concave mirror mounted 11 cm. below the optical axis and a small total reflection prism situated between the first microscope objective and the plate. By this means a brilliant image of the spectrum enlarged 20 times is projected on the screen, and the part to be analysed can readily be located and brought into position for measurement. When this is done the small prism is moved to one side and the primary optical system automatically comes into play.

Focussing and collimating arrangements are very simple. To focus, the first slit is opened wide and the second microscope objective is adjusted till the silver grains are brought to sharp focus on the screen. The first slit is then closed to the width required for analysing the spectrum, and by a motion of the first microscope objective the image of the slit is brought into sharp focus on the screen. To collimate, the unit of first slit and microscope objective is moved horizontally by means of a screw adjustment till a maximum galvanometer deflection is obtained, indicating coincidence with the second slit. Both first and second slits have rotation adjustments.

It will be noted that there is a change from the old direct reading instrument in that a double-slit system is employed. The use of two slits has the advantage of minimising effects due to scattering of light in the photographic emulsion and to stray light in the lens system. Such effects are ordinarily of little consequence, but in the photometry of very narrow absorption lines and in the measurement of pure emission line spectra they may lead to appreciable errors. The double-slit system is less convenient to use than the single-slit arrangement which made possible a clear view of the spectrum while recording was in progress, but the provision of an auxiliary system of illumination has largely made up for any inconvenience caused by the change.

The instrument has been designed to make simultaneous records of the star spectrum and of the two sets of iron comparison lines. With this end in view both first and second slits are of sufficient length to include the star spectrum and an appreciable length of the comparison lines, while the three corresponding beams of light are separated by suitable masks placed in front of the second slit. The beam from the star spectrum passes straight through a light proof tube to the main photo-electric cell, located 50 cm. behind the second slit. The two beams from the comparison spectra are intercepted by small total reflection prisms and after a series of reflections pass through a second light proof tube to two secondary cells situated beside the main cell.

The Micrometer.—The base of the micrometer is a heavy iron casting with horizontal ways machined on its upper surface. The sliding carriage is a right-angled casting of aluminium. On the lower horizontal face of this casting are two brass strips accurately machined and ground, which slide on the cast-iron ways. On the vertical face of the casting a brass plate slides vertically in machined ways. The motion is provided by a screw of 1 mm. pitch fitted with a micrometer head and linear scale. Mounted on the plate is a turn-table of 20 cm. diameter on which the photographic plate is held by steel clips. The turn-table rotates through 360°, a feature which has added very greatly to the convenience of operation of the instrument. The area which can be covered without moving the plate is 6×9 cm. If overlapping is allowed, a plate 15 cm. wide and any ordinary length can be taken without cutting.

The main screw, made by the Gaertner Co., is of diameter 19 mm. with a 2-mm. pitch and a travel of 86 mm. The nut is attached to the sliding carriage, while the thrust is taken by a ball on the end of the screw which works on a hardened steel plate mounted on the iron base. Slack is taken up by means of a weight and cord. The screw has a shaft 22 cm. long which extends through the recording camera. On the end of the shaft is a micrometer head of 10 cm. diameter with 500 divisions etched on its circumference. A linear scale is mounted on top of the camera.

Integral with the micrometer head and of slightly smaller diameter is a pulley which takes the drive, by means of a belt, from an electric motor and speed reducer (900–1). A clutch on the pulley of the speed reducer makes it unnecessary to remove the belt when the micrometer is moved by hand. The maximum speed of the drive is 4 mm. per minute, while smaller speeds are obtained by the use of a rheostat in series with the motor.

The Recording Camera.—The recording device in general design somewhat resembles a motion picture camera. The sensitive paper passes from the supply spool (which has a capacity of 30 meters—100 ft.—of paper) to the recording drum, an aluminium cylinder of 63·6 mm. diameter and 127 mm. long. The recording drum is mounted directly on the micrometer shaft by means of bronze cylindrical bearings. After passing around the recording drum the paper goes to the lower spool containing the exposed supply. Directly below the recording drum is a second aluminium cylinder of precisely similar diameter mounted on roller bearings. The lower drum

is held against the recording drum by means of a spring tension of sufficient strength to prevent any slipping of the paper. The paper is kept taut by means of a light brake on the supply spool. The spool containing the exposed supply is driven by a slipping belt from the recording drum which automatically takes up any slack in the exposed paper. Waste of paper is minimised by a tape passing from the exposed spool to the recording drum, with a clip on the end for gripping the paper when starting.

Two magnifications are provided, × 100 and × 20. To use the 100-fold magnification the recording drum is directly connected to the shaft of the micrometer screw by means of a gear clutch so that they move with the same angular velocity. For the 20-fold magnification the recording drum and the micrometer screw shaft are connected by means of a set of four gears similar in arrangement to those which govern the relative motions of minute and hour hand in an ordinary clock. The change from × 100 to × 20 is effected by declutching and moving a gear shift lever.

The beam of light from the galvanometer mirror is admitted to the camera in the usual way by means of a long slit in contact with the periphery of the recording drum. In front of this slit is a cylindrical lens which has etched on its inner surface a set of lines 1 mm. apart with every fifth line accentuated. By allowing a general illumination to fall on the lens there is produced a grey background with a set of white horizontal lines forming a millimeter scale. Superposed on this scale is the black tracing due to the beam of light from the galvanometer mirror. Vertical marks recording every half revolution of the screw (corresponding to 1 mm. on the plate for the × 100 magnification) are impressed on the paper by light shining through three longitudinal slots cut in the lower drum. A removable oblong window above the recording drum makes it possible to write the number or other information concerning the plate on the sensitive paper. The cover of the slit and cylindrical lens is painted white and has a millimeter scale for determining the position of the galvanometer spots before starting or, if necessary, for making actual measurements. A mirror is provided for watching the motion of the galvanometer spots while recording is in progress.

The Galvanometer Lamp.—A 250-watt 100-volt lamp mounted in a ventilated case of sheet-iron provides the source of light for the galvanometer beams. Three slits cut in a sliding brass plate provide illumination for the three separate galvanometers. Shutters are provided for eliminating the auxiliary beams when only the main galvanometer is in use.

The Electrical System.—The electrical system is illustrated in fig. 1. It consists of a single tube amplifier circuit with an auxiliary resistance connected to the galvanometer through a variable high resistance to balance out the plate current. This circuit has been described by Nottingham.* Its electrical characteristics can be varied through fairly wide limits by suitable changes in the plate potential and grid bias. The writer has used a No. 22 or No. 32 valve with a floating grid and a plate potential of 10 volts. This arrangement, used with a cæsium vacuum photo-electric cell and a Moll

* *Journ. Franklin Institute*, **209**, 287, 1930.

micro-galvanometer, gives a maximum deflection for ordinary slit widths of 1000 mm. from full light to darkness. Such a high sensitivity is unusable for most purposes so an Ayrton shunt is provided which varies the sensitivity of the galvanometer by five steps over a range of 1–16. The flexibility provided in this way is extremely useful in the measurement of negatives of different quality and density. Owing to the low voltages employed the amplifier is very stable and free from electrical difficulties, while the excellent quality of present-day commercial valves and cells makes it relatively inexpensive to construct. Two 90-ampere hour car batteries provide the source of potential for both valves and photo cells. As the maximum

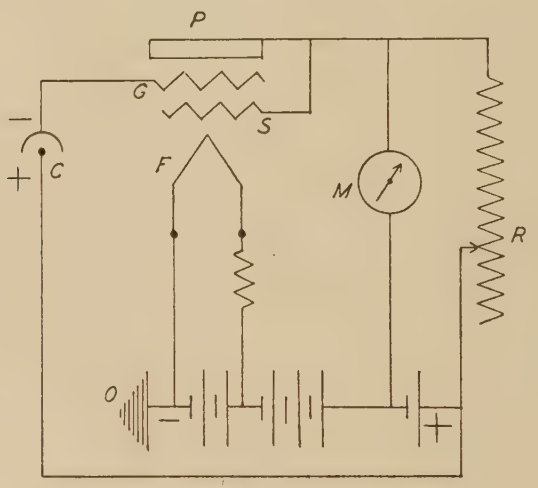

FIG. 1.—*Diagram of Valve Amplifier.*

F, valve filament. S, space charge grid. G, control grid. P, plate (P and S are directly connected). C, photo-cell. M, galvanometer (shunt omitted from diagram). R, variable resistance 50,000 *ohms. O, ground connection. Voltage of each cell,* 2 *volts.*

current drawn by a single amplifier (for lighting the filament) is of the order of 1/10 ampere it is probable that smaller batteries would serve the purpose equally well. In the present instrument all three amplifiers are operated from the same source of potential, each of the two auxiliary photo-electric cells having its own amplifier. For recording the positions of the iron comparison lines two modified Cambridge pot galvanometers have been used. Their sensitivity is about 1/10 that of the Moll instrument and their period about the same (0·20 sec.). They have proved to be very satisfactory, and, provided extreme sensitivity were not desired, one of these instruments could be used to replace the Moll galvanometer at a great saving in cost. It may also be noted that any ordinary galvanometer could be used with such an instrument, providing the plate is not run through too rapidly. The writer employed a galvanometer with a period of $1\frac{1}{2}$ sec. with complete satisfaction for a few months, while the Moll instrument was being repaired.

Mounting.—The instrument is mounted on a heavy concrete pier 200 cm. long with an L section 60 cm. long. The top of the pier is faced with a

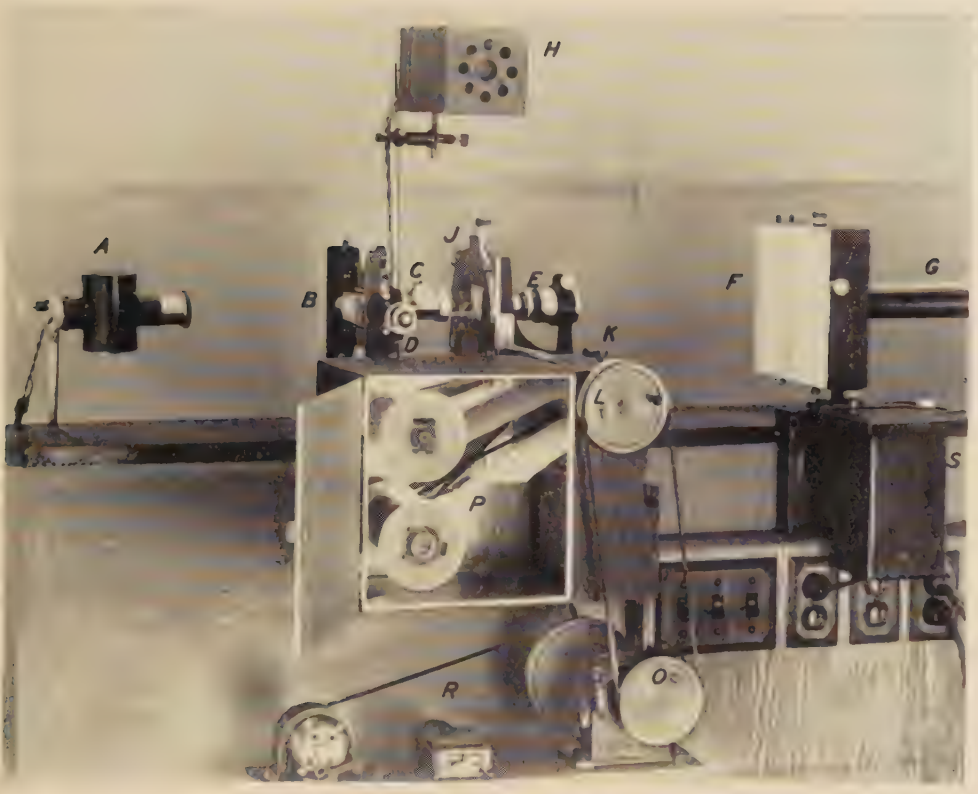

Fig. 2.—Optical System and Recording Device.

A, lamp housing and condenser. B, first slit. C, first microscope objective. D, screw motion for band C. E, second microscope objective. F, screen with second slit in centre. On the white face of the screen are shown the masks for limiting the beams. On the top and near side are the screw adjustments for altering the length and width of slit and the mirror adjustments for the auxiliary beam. G, light proof tubes to photo-cells. H, lamp housing for auxiliary source of light. The total reflection prism is shown in place between C and the plate, while the concave mirror is out of sight behind the camera box. J, micrometer carriage, showing vertical micrometer and turn-table. A plate is in position for measurement. K, top of gear casing and gear shift lever for altering magnification. L, clutch on micrometer head. O, clutch on speed reducer pulley. P, camera, showing supply spool (above), exposed spool (below), crossed belt for driving exposed roll and tape with clip for starting paper. Part of the aluminium recording drum is also visible. R, electric motor and speed reducer. S, galvanometer lamp.

cast-iron plate 2·5 cm. thick. The optical and mechanical parts, as well as the amplifiers (which are contained in a sheet-metal box), are mounted on the main pier, while the three galvanometers, also in a metal box, are on the L section. The batteries are contained in a shielded metal box beside the pier, while the necessary wires are led in a shielded conduit electrically bonded to the amplifier and galvanometer cases. Fig. 2 (Plate 9) gives an idea of the pier with the optical system and recording camera. The electrical parts are out of sight on the right of the figure.

Performance.—In use, the instrument has exceeded expectation. The working of the mechanical parts approaches perfection, while the electrical and optical properties are equally satisfactory. The large supply of paper, making possible the measurement of 10 or 20 plates without reloading or developing, is a great convenience, while the removal of short lengths of paper is easy and rapid. The change of magnification is effected in a few seconds, and the usual practice has been to use the large magnification for the star spectrum and change to the small for the calibration. Since in order to record the calibration it is usually necessary to rotate the plate through an angle of 90°, the provision of a turn-table on the micrometer carriage has resulted in much saving of time. As a simultaneous record of the iron lines can be made with little extra trouble the writer has recently followed the practice of putting them on each tracing. They may be used for measurements of wave-length or radial velocity and form a convenient guide for the analysis of a complex spectrum. Fig. 3 (Plate 10) shows typical tracings of the spectrum of the star α Cygni in the region of $H\gamma$ made with the 20-fold and the 100-fold magnification.

An estimate of the maximum resolving power has been made by a method similar in principle to that recently discussed by H. H. Plaskett.* The two slits were set at a width projected on the plate of 0·005 mm. A razor-blade was then substituted for the photographic plate and a trace was made by running the edge of the blade across the beam. The photo-electric cell was "loaded" with extra light in such a manner that the response to light passing through the optical system was approximately linear. The 100-fold magnification was used and film was substituted for sensitive paper to facilitate accurate measurement of the tracing. Measurement of the tracing with a micrometer machine gave the curve connecting distance moved by the blade with galvanometer deflection. Since with this arrangement a linear relationship exists between deflection and intensity this curve can be treated as a distance-intensity curve. By mechanical differentiation at a series of points along this curve a second curve is obtained representing the distribution of light affecting the photo-electric cell (analogous to Plaskett's slit pattern). The form of the second curve is such as to indicate a resolving power sufficient to separate well over 200 lines per mm. Integration of the second curve showed that approximately 90 per cent. of the light affecting the photo-electric cell was cut off by a motion of the blade equal to the projected slit width (0·005 mm.). A precise interpretation of this result in terms of the purity factor of the instrument is difficult owing to the use of two slits

* *M.N.*, 95, 160, 1934.

and to the presence of diffraction effects at the edge of the blade, but it is at least certain that the performance of the lenses is reasonably good and that the maximum resolving power is greater than would ever be required for the analysis of photographic images.

The following test was made of the accuracy of linear measurement possible with the ×100 magnification. Two parallel lines were ruled on a glass plate, and their distances apart measured with an accurate visual machine was found to be 1·440 mm. The plate was then set up in the recording photometer and separate tracings were made at ten different positions uniformly distributed along the length of the micrometer screw. Measurement of the ten tracings showed a maximum error of 0·002 mm.* in the determination of the separation, and an average error without regard to sign of 0·0008 mm. Similar tracings made with the ×20 magnification gave errors approximately five times the above values.

Dominion Astrophysical Observatory,
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OBSERVATIONS ON THE INTENSITY OF THE TOTAL RADIATION FROM SUNSPOTS AND FACULÆ.

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(Communicated by the Director, Solar Physics Observatory, Cambridge)

I. *Introduction*

Quantitative observations on the intensity of the radiation from sunspots are not very numerous. Many years ago Wilson † used a Boys radiomicrometer to measure the intensity of the radiation passing through a pinhole in a screen on which a large image of the Sun was thrown. He concluded that the ratio of the intensity of the radiation from a spot umbra to that from the surrounding photosphere varied, on the average, from about 0·4 near the centre of the disc to about 0·8 near the limb, while individual spots near the centre of the disc gave values as low as 0·3. Pettit and Nicholson ‡ have published the results of observations in which a thermocouple and galvanometer were used to record the intensity of the radiation passing through a small aperture in a screen as the whole apparatus was moved across the image of a spot. Observations on six spots gave values of the intensity ratio umbra/photosphere ranging from 0·43 to 0·52 with a mean value of 0·47. A rough estimate was made of the effect of scattered

* In the annual report of the work of this observatory (*M.N.*, **96**, 327, 1936) 0·002 mm. was erroneously given as the average error.

† W. E. Wilson, *Roy. Soc. Proc.*, **55**, 1894, and *M.N.*, **55**, 457, 1895.

‡ E. Pettit and S. B. Nicholson, *Ap. J.*, **71**, 153, 1930.

light by taking observations just outside the limb of the Sun. The value of 0·47 was confirmed by the results of an investigation on the distribution of energy in the sunspot spectrum. Their paper seems to imply that the value of this intensity is about the same for all spots.

A number of observers have made photometric observations using direct photographs of the Sun taken through filters of various colours. Such observations were made by Schwarzschild and Villiger * in the ultra-violet, and by Barabascheff and Semejkin † using a yellow filter; they concluded that the ratio of umbral intensity to that of the neighbouring photosphere increased from the centre of the disc to the limb. Richardson, ‡ however, in a recent similar investigation finds no evidence for any variation of this ratio over the disc, for sunspots. His observations include estimates of the intensity of faculæ which show a considerable intensity increase near the limb.

In a recent investigation by Wanders § a new technique is employed. A spectrum is photographed with the slit of the spectrograph set across a sunspot and employing special precautions to avoid astigmatism. The distribution of intensity across the spot and neighbouring photosphere is then investigated in various spectral regions. A very thorough investigation was made of the effect of light scattered by the sky and instruments, and a correction applied. He finds, in agreement with Richardson, that there is no evidence of a variation of the intensity ratio umbra to photosphere across the disc, and that any apparent effect may be ascribed to the increase in the effect of light scattered by the sky and instruments when the spot is foreshortened near the limb.

The importance of such investigations for our knowledge of the physical conditions in sunspots has been emphasised by Minnaert and Wanders. ‖ It would appear to be impossible on any reasonable assumptions to account for a constant intensity ratio umbra/photosphere if the visible region of the spot is in adiabatic equilibrium.

The observations discussed in this paper were obtained with a thermo-couple and galvanometer behind an aperture in a screen. The method has the advantage that it avoids completely the troubles of photographic photometry. It has the disadvantage that it takes several seconds to record the intensity distribution across a spot, and trouble from unsteadiness of the image is likely to be worse than in a method where a photograph is taken in a fraction of a second. The consistency of the results of successive trails across the same spot shows, however, that the method will yield reliable results under good observing conditions.

Fig. 1 (p. 738) shows, for example, two successive trails in opposite directions across a group containing a large spot and several small spots, on 1935 October 23. The two trails over the main spot are very similar.

* K. Schwarzschild and W. Villiger, *Ap. J.*, **23**, 345, 1906.
† N. Barabascheff and B. Semejkin, *Zeit. für Astrophys.*, **5**, 54, 1932.
‡ R. S. Richardson, *Ap. J.*, **78**, 359, 1933.
§ A. J. M. Wanders, *Zeit. für Astrophys.*, **10**, 15, 1935, and **8**, 108, 1934.
‖ M. Minnaert and A. J. M. Wanders, *Zeit. für Astrophys.*, **5**, 297, 1932.

II. *Experimental Details*

The solar image of approximately six inches diameter is formed by the McClean solar installation consisting of a cœlostat and horizontal telescope with a 12-inch apochromatic objective of 60 feet focal length. The image is thrown on a white cardboard screen pierced by a small pinhole of approximately 0·15 mm. diameter; radiation passing through the small aperture in the screen falls on a 2-inch microscope objective, which forms a reduced image of the aperture on the blackened junction of a Moll "quick vacuum thermocouple" connected directly to a Moll micro-galvanometer. The

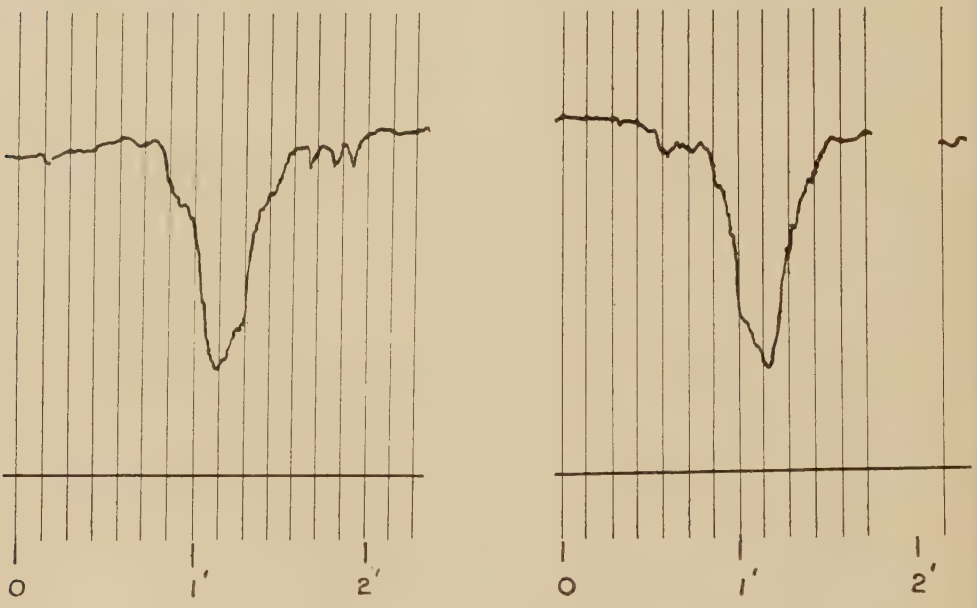

Fig. 1.—*Spot F*, 1935 *October* 23.

deflection of the galvanometer is recorded on bromide paper on a rotating drum driven by clockwork at a speed which is adjustable over a considerable range. The chief factor influencing the choice of the recording apparatus was its speed of response. Observations may be obtained rapidly while atmospheric conditions vary as little as possible. When the magnetic field of the galvanometer is adjusted to give critical damping, the final deflection is reached within 0·4 second of the radiation falling on the thermocouple.

By means of the slow motions of the cœlostat, a sunspot umbra or any other selected point on the disc may be set on the aperture in the screen. To study the distribution of intensity across a disturbed area on the Sun the solar image is drifted slowly across the aperture. For this purpose the secondary mirror of the cœlostat may be rotated about a vertical axis with a definite angular velocity by a synchronous electric motor driving through suitable gearing. The velocity and direction of motion of the image on the screen are then known.* The motor is reversible, and is controlled

* T. W. Wormell, *M.N.*, **96**, 159, 1936.

from the image end of the telescope. The area which is being investigated may thus be drifted backwards and forwards across the aperture of the screen, and its position is under complete control. Time marks are impressed on the record of galvanometer deflections by a simple device actuated by a second small synchronous motor. The time interval between two such marks is 2·70 seconds. The exact value of the drift of the image during this time depends on the Sun's declination; during the summer it is approximately 10 seconds of arc.

The linearity of the scale of the thermocouple has been tested in two ways. First, the two halves of the objective were screened off alternately, when it was found that the sum of the galvanometer deflections in the two cases was sensibly equal to the deflection with full aperture. A further check was made by interposing in the beam at various points of its path a previously calibrated metal gauze screen. Again no sensible departure from linearity was detected.

Another possible source of error is that in the infra-red the image may be appreciably out of focus. The exact magnitude of the error introduced in this way is unknown, but it should be largely included in the correction for scattering considered below.

The most obvious difficulty in the observations is the effect of light from the photosphere scattered into the image of the spot by the Earth's atmosphere and the various surfaces of the optical train. An attempt is made to estimate the magnitude of this effect by recording the distribution of intensity when the limb of the solar image is drifted radially across the screen aperture. Such records are made during the observations on every day when weather conditions permit. The method of correcting for scattering is considered in detail in the next section.

A further difficulty arises from the fact that the absorption of light by the atmosphere and the instruments is selective. Since the distribution of energy with wave-length is different for radiation from the photosphere and from a spot, selective absorption changes the ratio of the total energies which reach the measuring apparatus in the two cases from its value outside the Earth's atmosphere. If the difference between the qualities of two radiations were too great, direct observations on the total energy of the radiations after traversing the atmosphere and an optical system would be meaningless. In the present case, however, the energies are not in widely different parts of the spectrum; according to Pettit and Nicholson,* for example, the maximum ordinate in the spectral energy-curve of a sunspot is only about 400 angstroms further towards the red than the corresponding maximum in the curve for the centre of the photosphere.

In order to estimate the magnitude of this source of error it was assumed, as an approximation, that the distribution of intensity with wave-length corresponds in the case of spot and photosphere to that from black bodies at 4500° K. and 6000° K. respectively. The ratio of the total energies measured by a perfectly black-receiving instrument outside the Earth's atmosphere would then be 0·316. The radiation traverses the Earth's

* *Loc. cit.*

atmosphere, is twice reflected from a silver surface, and finally passes through 5·2 cm. of glass before falling upon the thermocouple junction. The spectral energy-curves at these two temperatures were computed and multiplied, ordinate by ordinate, by the atmospheric transmission for a place near sea-level. Values were computed at 14 wave-lengths ; the mean of the atmospheric transmission cœfficients for Potsdam * and Washington † were used for the general absorption, and an estimate of the average effect of the deep atmospheric bands in the infra-red was included. The calculations were carried out for transmission through air mass 1 and through air mass 2. The resulting figures were next multiplied twice by the reflection cœfficients of silver ‡ at the various wave-lengths, and finally by the transmission cœfficients for 5·2 cm. of glass using data given by Hovestadt. § The values used were the means of those for ordinary specimens of crown and light flint glass ; the difference between the results, if the glass is assumed to be all crown, or all flint, is barely perceptible. The final distributions of intensity with wave-length were plotted and the areas under the curves determined graphically. The results were as follows :—

Ratio of total energies before entering the Earth's atmosphere .	.	0·316
Ratio of total energies reaching the thermocouple { Air mass 1 .	.	0·34
Ratio of total energies reaching the thermocouple { Air mass 2 .	.	0·36

It remains to consider any selective effect of the blackened receiving surface of the thermocouple itself. A rough test with a small dispersion solar spectrum showed no evidence of any selective effect in the visible and near infra-red region. It is known, moreover, that the selective effect of a suitably blackened surface on the solar radiation is very small ; the determination of the solar energy-curve by the Smithsonian observers depends indeed on this assumption.|| It is clear, too, from the above discussion that any selective effect would have to be of considerable magnitude to affect the results of this paper appreciably.

It appears, therefore, that the observed values of the ratio of the intensity of the sunspot umbra to that of the surrounding photosphere will be systematically too high by an amount varying from about 7 per cent., if observations could be made with the Sun in the zenith, to about 12 per cent. if observations are made through air mass 2 (*i.e.* zenith distance about 60°). The observations discussed in this paper were all made through an air mass less than 2, and the great majority through an air mass less than 1·5.

The estimate of the magnitude of the effect of selective absorption is supported by the results of Pettit and Nicholson at Mount Wilson. They found, for the spots they observed, a value of 0·445 for the average ratio of the total radiation emitted from the spot to that from the surrounding photosphere, using the observed distribution of energy with wave-length. A direct observation of the total energy with a thermocouple gave a mean value of 0·47.

* J. Wilsing, *Potsdam Publ.*, **72**, 69, 1917.
† *Smithsonian Annals*, **3**, 135, 1918.
‡ Hagen and Rubens, *Ann. der Physik*, **8**, 1, 1902.
§ *Jena Glass*, Macmillan & Co., 1902.
|| Abbot, Fowle and Aldrich, *Smithsonian Miscellaneous Collections*, **74**, 7, 1923.

III. *The Scattering Correction*

The problem of correcting the observed value of the intensity of the radiation from a sunspot umbra for the light from the surrounding photosphere scattered into its image has been discussed in detail by Wanders.* The observational data available for estimating the magnitude of the scattering consist of a record of the apparent intensity of the radiation from points on the solar disc in the vicinity of the limb, both inside and outside the true position of the limb. Scattering here includes the averaged effect of unsteadiness of the image.

The apparent intensity just outside the limb falls off rapidly until it reaches a small value, of the order of 2 or 3 per cent. of the intensity at the limb; beyond this point the intensity of the background or bright sky falls off very slowly, the diminution being small even in a distance comparable with a solar diameter. The value of this background is first subtracted from all observed intensities. Throughout the remainder of this paper it is assumed that the value of the background sky has been removed from the observed intensities. We turn next to the main scattering effect. Following Wanders, but simplifying the procedure to some extent, we assume that as a result of all the scattering experienced in traversing the Earth's atmosphere and the optical train of the observing apparatus, a luminous point outside the atmosphere gives in the focal plane of the telescope a distribution of intensity $\frac{a}{\pi}e^{-ar^2}$, where r is the distance from the centre of the image and a is a scattering parameter whose value at any moment is a measure of the amount of scattering. Wanders has shown that such a simple assumption gives an adequate explanation of the observed distribution of intensity near the limb of the solar image. In order to compute the apparent distribution for a given value of a we clearly require to know the true distribution of intensity near the limb. While the precise form of this distribution is uncertain, there is little doubt that, to the accuracy required here, there is a sudden discontinuous jump from zero to a finite value as we cross the limb.† Inside the limb the intensity increases as we travel farther in, but for our present purpose the precise form of this increase is unimportant, as the apparent intensity distribution just outside the limb is not sensitive to the exact form of the true variation inside the limb. We therefore compute the scattered intensity near the limb by assuming that the true distribution is a sudden jump from intensity zero to intensity I at the limb, and a constant intensity I inside the image. It is readily seen that if $i(x)$ be the apparent intensity at a distance x seconds outside the limb,

$$i(x) = I \cdot \frac{1}{\sqrt{\pi}} \int_{x\sqrt{a}}^{\infty} e^{-u^2} du = \tfrac{1}{2} I\{1 - Erf(x\sqrt{a})\}. \qquad (1)$$

We notice that when $x = 0$, $i(x) = \frac{1}{2}I$.

* *Loc. cit.*

† M. Minnaert, *M.N.*, **89**, 197, 1928.

We may use (1) to determine a from the observed distribution of $i(x)$ with x. We assume that the observed value of $i(x)$, 5 seconds of arc within the limb, gives a sufficient approximation to the true value of I; the curves given in the paper of Wanders are a sufficient justification of this, for all reasonable values of a. The value of $i(x)$ is then measured 5″ outside the limb, and hence the value of a is read off from a curve of $\frac{i(5'')}{I}$ against a, based on equation (1).

We turn next to the estimation of the scattering into a spot umbra once a has been determined. We consider an idealised spot in which the true intensity is constant across the spot. Two cases are especially simple. First, a long narrow spot of width $2b$. It is readily seen that the intensity of the scattered light at the centre of the spot is

$$\frac{2I}{\sqrt{\pi}}\int_{b\sqrt{a}}^{\infty} e^{-u^2}du = I(1 - Erf(b\sqrt{a})), \tag{2}$$

where I is now the difference of intensity between the spot and the photosphere. The amount of scattered light is, in this case, just twice the intensity at a distance b outside the limb of a solar image of true intensity I at the limb. Secondly, consider the case of a circular spot, of radius b. It is readily seen that in this case the scattered intensity at the centre is

$$Ie^{-ab^2}. \tag{3}$$

In the case of a real spot, when we attempt to estimate the scatter into the umbra the problem is complicated by the presence of the penumbra. Calculations show, however, that unless the scattered intensity is large, when in any case the observations have little value, the majority of the scattered light at the centre of the umbra comes from the inner edge of the penumbra, and the intensity therefore depends on the difference of intensity between the umbra and the inner edge of the penumbra. This contrast can usually be determined fairly reliably from the records, and the scattering is computed from its observed value using formula (2) or (3) whichever approximates best to the shape of the spot. If sI be the scattering computed from (2) or (3), and h the observed contrast between umbra and penumbra, the true contrast will be $\frac{h}{1-s}$, and the observed depth of the umbra is to be increased by an amount $h\left\{\frac{1}{1-s} - 1\right\}$.

IV. *The Distribution of Radiation Intensity across a Sunspot*

We consider, as a typical case of a fairly large spot near the centre of the disc, the records of the spot of 1935 July 21, fig. 2. The diameter of the penumbra was about 40″ and of the umbra 14″, the spot being nearly circular. The observed apparent intensity falls regularly from the photosphere, whose intensity is taken as unity, through the penumbra to a value of 0·75 at the

inner edge of the penumbra, and then drops sharply to the value of 0·31 at the centre of the umbra. It must be pointed out, however, that such observations are seriously affected by the scattered light present, even when the scattered light at the centre of the umbra is extremely small. When this is taken into account the observations may be completely accounted

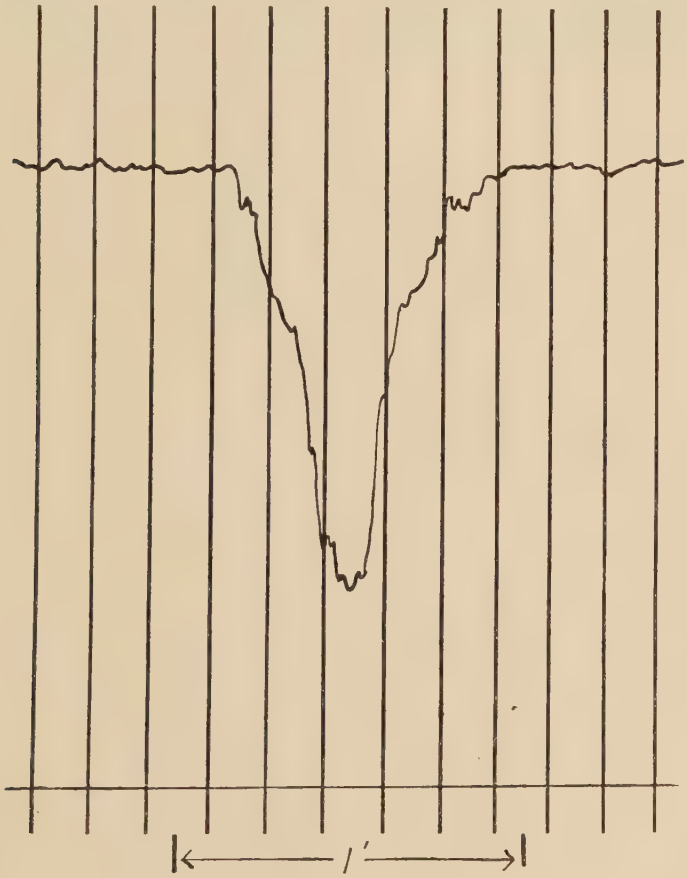

FIG. 2.—*Spot B*, 1935 *July* 21.

for by an umbra of constant intensity 0·30, of diameter 10″, surrounded by a penumbra of constant intensity 0·80 and diameter 30″.

In fig. 3 (p. 744) the dotted line represents the computed distribution of intensity that would be observed for a spot of these dimensions, assuming the true distribution suggested above; the calculations were made, for simplicity, for a long spot of the width shown (the general effect would be similar for a circular spot), assuming the scattering law of the last paragraph, with a value of $a = 0{\cdot}065$, which corresponds to good observing conditions. The computed curve is superposed on the observed curve of fig. 2.

It is clear that, for spots of this size, such observations cannot give accurate information as to the true distribution of intensity across the umbra and penumbra. They do, however, give a reliable measurement

of the intensity at the centre of the umbra and an approximate value of the intensity in the penumbra. Observations on a very large spot would, however, give more detailed information.

V. *The Intensity of the Radiation from the Centre of a Sunspot Umbra*

The observations are tabulated in Table I. Each individual spot has a letter attached to it, for identification, in column 1 ; column 2 gives the

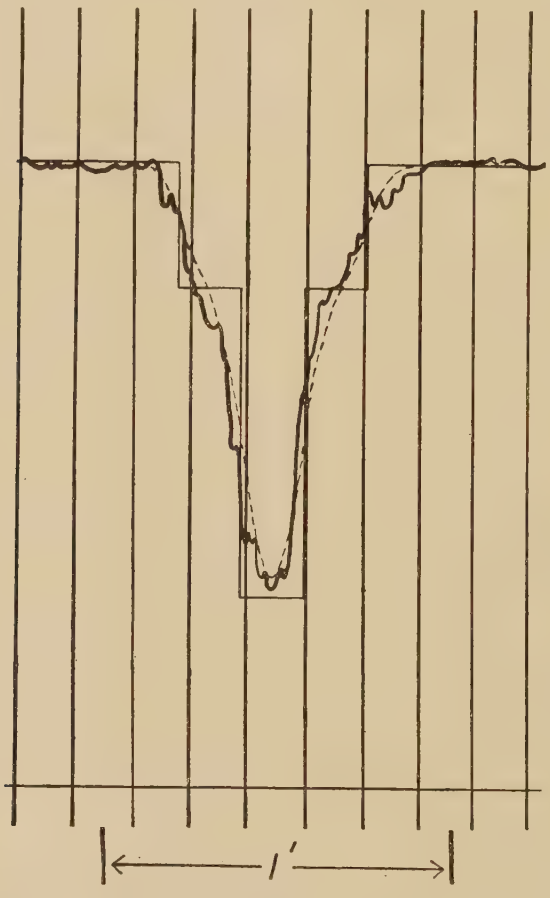

Fig. 3.

date of the observations, column 3 the apparent intensity at the centre of the umbra, measured direct from the record ; column 4 the value after correction for scattering, the probable error is indicated to show the scatter of the individual observations ; column 5 gives the number of independent records from which the values in the two previous columns were deduced ; column 6 the apparent distance of the spot from the centre of the disc expressed as a fraction of the Sun's radius ; column 7 a rough estimate of the apparent size of the spot umbra ; column 8 the value of the scatter parameter *a*, deduced from observation at the limb.

TABLE I

Sunspots

	Date	Central Intensity of Umbra		Number of Independent Observations	d/R	Projected Area of Umbra in Square Seconds of Arc	Scattering Parameter, a
		Uncorrected	Corrected for Scattered Light				
A	22 : 5 : 34	0·54	0·52	1	0·50	140	...
B	16 : 6 : 34	0·54	0·47 ± ·03	2	0·85	80	...
B	17 : 6 : 34	0·51	0·45 ± ·01	3	0·70	130	...
C	10 : 7 : 34	0·52	0·46 ± ·02	2	0·62	100	...
C	11 : 7 : 34	0·57	0·52	1	0·48	100	...
C	17 : 7 : 34	0·65	0·61 ± ·03	2	0·86	60	...
A	8 : 6 : 35	0·49	0·28 ± ·03	5	0·70	40	0·066
B	21 : 7 : 35	0·31	0·30 ± ·00	3	0·44	160	0·065
B	23 : 7 : 35	0·35	0·32 ± ·01	2	0·45	150	0·061
B^1	24 : 7 : 35	0·52	0·39 ± ·01	2	0·57	100	0·030
B^1	25 : 7 : 35	0·63 0·57	0·40 ± ·05	3	0·70	100	0·013 0·034
B^1	26 : 7 : 35	0·68	0·39 ± ·04	4	0·83	60	0·010
C_1	31 : 7 : 35	0·82	0·66 ± ·05	6	0·89	30	0·040
C_2	31 : 7 : 35	0·90	0·69 ± ·03	3	0·93	10	
C_1	1 : 8 : 35	0·72	0·49 ± ·01	3	0·76	40	0·006
C_1	2 : 8 : 35	0·70	0·40 ± ·02	5	0·61	50	...
C_1	5 : 8 : 35	0·78	0·72 ± ·02	3	0·42	30	0·043
C_1	6 : 8 : 35	0·87	0·77 ± ·03	3	0·54	diffuse group	0·038
D	5 : 8 : 35	0·70	0·51 ± ·03	4	0·93	40	0·043
D	6 : 8 : 35	0·60	0·52 ± ·05	3	0·84	70	0·038
D	7 : 8 : 35	0·55	(0·27 ± ·05)	6	0·71	70	0·021
D	8 : 8 : 35	0·53	0·43 ± ·03	6	0·58	60	0·065
D	10 : 8 : 35	0·49	0·36 ± ·02	7	0·40	70	0·046
D	13 : 8 : 35	0·68	0·59 ± ·01	4	0·71	70	0·043
E	10 : 9 : 35	0·68	0·62 ± ·02	3	0·97	40	0·070
F	21 : 10 : 35	0·42	0·28 ± ·01	3	0·32	200	0·017
F	23 : 10 : 35	0·31	0·30 ± ·01	5	0·63	300	0·044
F	25 : 10 : 35	0·40	0·32 ± ·02	8	0·89	160	0·043

It will be observed that in some cases the scattering correction is rather large; some of the observations were made under poor conditions with the double object of following the history of a spot in as much detail as possible and of testing whether reliable measurements could be made under such conditions.

There are some further points of interest in the table. Spot B 1935 shows a big increase in the apparent brightness of the umbra (column 3) between July 23 and 24. Between these two series of observations the fairly

large spot which had been observed on the 23rd broke up into a stream of small spots. This caused a great increase in the error due to the scattered light in the subsequent observations, and, after applying the scattering correction, the difference between the umbral intensities on the 23rd and 24th is much reduced.

Spot C 1935 shows a considerable real increase in brightness in the umbra between August 2 and 5—the spot was dying out. In the history of Spot D 1935 there is a very low value in column 4 on August 7. It appears probable that here the large scattering correction has been overestimated. There is, however, some evidence for a real variation in brightness; the uncorrected values were almost the same on August 7 and 8,

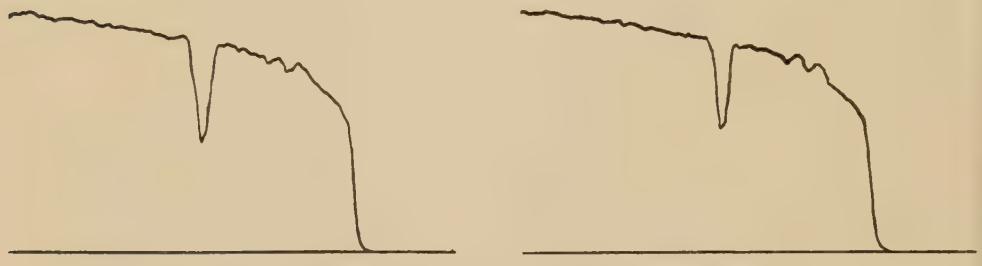

FIG. 4.—*Small spot and faculæ near the limb.* 1934 *June* 16.

but the scattering was very much greater on the 7th. It would seem, therefore, that the spot was really blacker on the 7th than on the 8th.

It would appear from Table I that even after a careful correction for the effects of atmospheric scattering there is a real variation of the central intensities of the umbræ among the individual spots, the larger spots tending to be darker. The same conclusion has been reached by Wanders.*

The two most reliable series of observations, on July 21 and October 23 (reproduced in figs. 1 and 2), when the effect of scattering was very small, agree in giving for the intensity from the umbra of a fairly large spot on the central part of the disc an intensity 0·30 that of the photosphere. A correction of 10 per cent. for the effect of selective absorption reduces this value to 0·27, which corresponds to an effective temperature of 4300° K. for the sunspot umbra viewed normally, assuming an effective temperature of 5950° K. for the photosphere near the centre of the disc. This is in excellent agreement with the results of Wanders * for spots of comparable size.

A discussion of the dependence of the intensity ratio umbra/photosphere on the position of the spot on the disc is rendered difficult in that the correction for selective absorption in the atmosphere is itself a function of that position. Any apparent systematic variation with position is very slight and the results cannot be considered as being in contradiction with the recent conclusions of Richardson † and Wanders * that the intensity ratio of umbra to photosphere is independent of position on the disc.

* A. J. M. Wanders, *Zeit. für Astrophys.*, **10**, 15, 1935. † *Loc. cit.*

VI. *Observations on Faculæ*

The traces obtained have been examined for areas showing an intensity of radiation greater than the normal intensity of the photosphere. For this purpose, whenever conditions permitted, records were made of trails to north and south of the spot as well as the trails across the spot umbra. A few records were also made across areas where there were no spots but

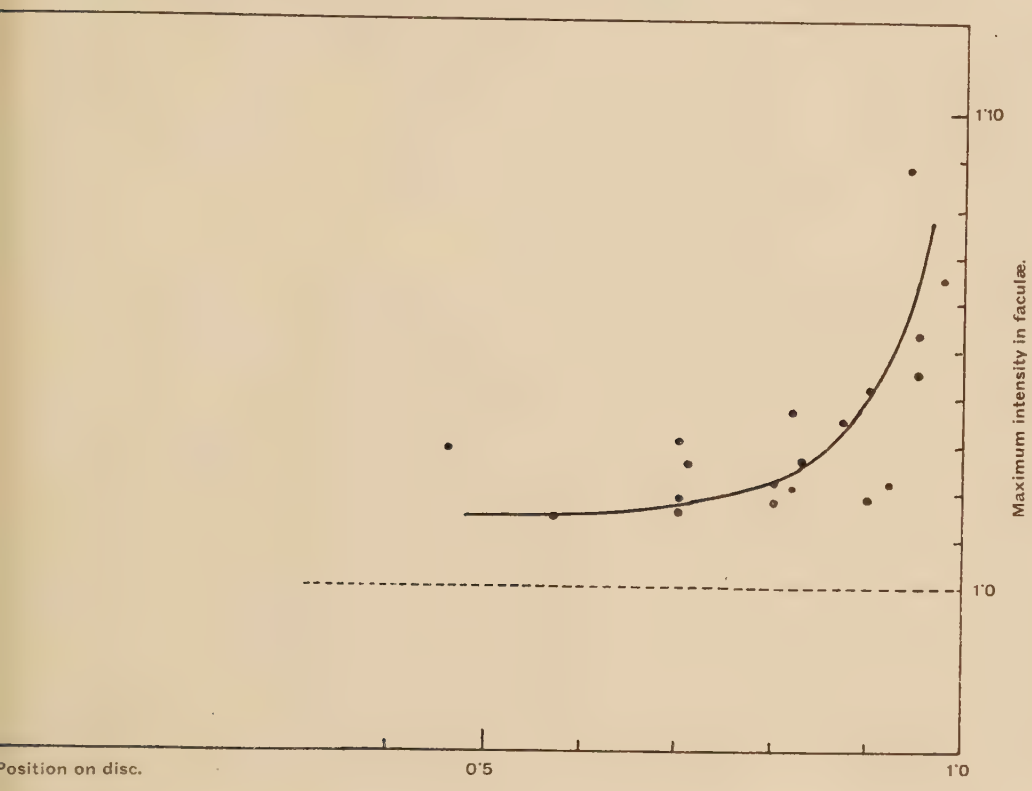

Fig. 5.

faculæ could be seen in the image. Fig. 4 is an example of a record with two successive trails showing a small spot and a well-defined patch of faculæ near the limb. The results where there was definite evidence of an area emitting excess of radiation are given in Table II together with the apparent position of the area on the disc. No attempt has been made to correct for scattering in this case—the correction in any case would be very small. The results are also plotted in fig. 5. The curve gives in effect the maximum intensity observed for faculæ as a function of d/R—that is, of sin θ, where θ is the angle between the line of sight and the normal to the solar surface. Faculæ cannot be detected by this method when d/R is less than about 0·5. For greater values the intensity increases to a mean value of about 1·10 that of the photosphere when very close to the limb.

TABLE II

Faculæ

	Date	Maximum Intensity in Facula	d/R		Date	Maximum Intensity in Facula	d/R
1	16 : 6 : 34	1·088	0·94	7	7 : 8 : 35	1·019	0·70
1	17 : 6 : 34	1·037	0·82	7	8 : 8 : 35	1·027	0·83
2	8 : 6 : 35	1·031	0·70	7[1]	2 : 8 : 35	1·016	0·70
3	23 : 7 : 35	1·029	0·46	8	6 : 8 : 35	1·022	0·92
4	24 : 7 : 35	1·026	0·71	9	,,	1·053	0·95
5	25 : 7 : 35	1·022	0·80	10	8 : 8 : 35	1·015	0·57
5	26 : 7 : 35	1·035	0·87	11	,,	1·021	0·83
6	25 : 7 : 35	1·065	0·98	12	10 : 8 : 35	1·042	0·90
7	31 : 7 : 35	1·019	0·90	13	10 : 9 : 35	1·045	0·95
7	1 : 8 : 35	1·018	0·80				

When a disturbed area containing spots and faculæ is near the central part of the disc, $\theta < 30°$ say, the maximum observed excess radiation from a patch of faculæ is only about 0·01 of the normal intensity of the photosphere. There will thus be a defect of radiation from the area unless faculæ of the above intensity cover an area of the order of 100 times the area of the spot umbræ. There was no sign in these observations of bright patches of such dimensions. The areas of exceptional brightness which were detected were small, amounting at most to perhaps 10 or 20 times the area of the spot umbræ. As far as observations of this type can show, it would appear then that a disturbed area of sunspots and faculæ when near the centre of the disc, *i.e.* when viewed approximately normally to the solar surface, is emitting less energy than the normal quantity. (It must be remembered, however, that a small excess of radiation over a very large area of the solar surface would escape detection by such a method which depends essentially on observing abrupt variations of brightness.)

When a disturbed area is near the limb, conditions are considerably different. The effect of foreshortening in diminishing the effective radiating area will be the same on the average for spots and faculæ. The intensity of the spot umbra is sensibly the same fraction of the photospheric intensity as it was near the centre of the disc, but the intensity of excess radiation of faculæ has risen to something like 0·10 of the photosphere. The total radiation from the area of disturbance is thus likely to exceed that from a similarly placed undisturbed area of the photosphere. We are thus led to anticipate a variation in the total radiation from the Sun as an area of disturbance travels across the disc, the radiation being below its normal value when the disturbance was near the centre of the disc, and perhaps above its normal value when the disturbance was close to the limb. Such a variation could scarcely be detected by the ordinary methods of determining the solar constant. It requires a very big spot to cause a diminution of 10^{-3} of the total solar radiation when near the centre of the disc (apart from

a possible partial compensation by faculæ). It is perhaps just conceivable that a recurrence tendency related to the solar rotation period might be introduced into geophysical phenomena. Such a tendency is, of course, familiar in terrestrial magnetism, but is presumably caused by a solar phenomena which varies through a much greater range than that of the total radiation.

It is a pleasure to express my gratitude to the Director, Professor F. J. M. Stratton, for his continued interest and encouragement during the course of these observations, and also to the Astronomer Royal and Mr. H. W. Newton of the Royal Observatory for the supply of prints and measurements from the Greenwich photoheliograms.

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REMARKS ON SOME THEOREMS IN THE DYNAMICS OF A STEADY STELLAR SYSTEM.

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1. Among certain theorems in this branch of Dynamics two of the most noteworthy are the Jeans * and Eddington † theorems. We shall state these.

Jeans's Theorem.—The general integral of the fundamental equation of Stellar Dynamics, viz.

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + X\frac{\partial f}{\partial u} + Y\frac{\partial f}{\partial v} + Z\frac{\partial f}{\partial w} = 0, \qquad (1)$$

where (x, y, z) are cartesian co-ordinates, (u, v, w) the corresponding velocities, (X, Y, Z) the force per unit mass at (x, y, z) and f is the distribution function at (x, y, z) at time t,

is an arbitrary function of the first integrals of motion of an individual star at (x, y, z), the equations of motion being of course

$$\frac{du}{dt} = X, \qquad \frac{dv}{dt} = Y, \qquad \frac{dw}{dt} = Z. \qquad (2)$$

Eddington's Theorem.—In a steady stellar system, in which the distribution of velocities is of the Schwarzschild's ellipsoidal type, the *principal velocity-surfaces*, defined to be the triply orthogonal system of surfaces whose normals at every point are parallel to the principal axes of Schwarzschild's ellipsoid at that point, are confocal quadrics.

* J. H. Jeans, "On the Theory of Star-streaming and the Structure of the Universe," *M.N.*, **76**, 70, 1915.

† A. S. Eddington, "Dynamics of a Stellar System (III)," *M.N.*, **76**, 37, 1915.

Jeans's theorem is the more general of the two, while Eddington's theorem is more important from the practical point of view, since the distribution of velocities in the local universe is known to be of the Schwarzschild ellipsoidal type. Eddington published his theorem before Jeans published his. It is not surprising, therefore, that Eddington's proof of his theorem is independent of Jeans's theorem, and is derived from first principles using the fundamental equation of Stellar Dynamics in the system of co-ordinates defined by the "principal velocity surfaces" forming a triply orthogonal system. Although Eddington proved his theorem for the case in which the distribution function is the exponential one, the theorem can be extended to the general distribution function of the ellipsoidal type. For, following Eddington's notation let the number of stars in the spatio-velocity element $Pd\lambda \,.\, Qd\mu \,.\, Rd\nu \,.\, dudvdw$ be $Pd\lambda Qd\mu Rd\nu \,.\, f(-a^2u^2-b^2v^2-c^2w^2+k)dudvdw$ and let σ be the star-density. Then

$$\sigma \equiv \iiint_{-\infty}^{+\infty} f(-a^2u^2-b^2v^2-c^2w^2+k)dudvdw, \tag{3}$$

which now replaces the equation $abc\sigma = e^{\kappa}$ of Eddington's paper. Jeans's theorem now gives that $\psi \equiv a^2u^2+b^2v^2+c^2w^2-K$ must be a first integral or a function of the first integrals of the equations of motion.

Let p_1, p_2, p_3 denote the "generalised momenta" corresponding to the co-ordinates (λ, μ, ν). Then the Hamiltonian H of the motion of a star is

$$H = \tfrac{1}{2}(p_1^2/P^2 + p_2^2/Q^2 + p_3^2/R^2) - \phi.$$

For ψ to be a function of the first integrals of motion, the "Poisson Bracket" (ψH) must vanish, where

$$(\psi, H) = \frac{\partial(\psi, H)}{\partial(p_1, \lambda)} + \frac{\partial(\psi, H)}{\partial(p_2, \mu)} + \frac{\partial(\psi, H)}{\partial(p_3, \nu)},$$

the usual Jacobian notation being used.

Thus

$$\begin{vmatrix} \dfrac{2a^2p_1}{P^2}, & p_1^2\dfrac{\partial}{\partial\lambda}\left(\dfrac{a^2}{P^2}\right) + p_2^2\dfrac{\partial}{\partial\lambda}\left(\dfrac{b^2}{Q^2}\right) + p_3^2\dfrac{\partial}{\partial\lambda}\left(\dfrac{c^2}{R^2}\right) - \dfrac{\partial k}{\partial\lambda} \\ p_1/P^2, & \dfrac{p_1^2}{2}\cdot\dfrac{\partial}{\partial\lambda}\left(\dfrac{1}{P^2}\right) + \dfrac{p_2^2}{2}\cdot\dfrac{\partial}{\partial\lambda}\left(\dfrac{1}{Q^2}\right) + \dfrac{p_3^2}{2}\dfrac{\partial}{\partial\lambda}\left(\dfrac{1}{R^2}\right) - \dfrac{\partial\phi}{\partial\lambda} \end{vmatrix} + \ldots + \ldots = 0.$$

And now equating the coefficients of the various terms in the p's, we obtain equations (13) of Eddington's paper. These equations together with the Lamé relations satisfied by P, Q, R yield, on integration for P, Q, R, the final result, as in Eddington's paper.

Jeans's theorem is of the utmost importance by reason of its generality. Prior to publishing his theorem, Eddington published * two papers on

* A. S. Eddington, "The Dynamics of a Globular Stellar System (I)," *M.N.*, **74**, 5, 1913; "The Dynamics of a Globular Stellar System (II)," *M.N.*, **75**, 366, 1914.

the dynamics of a globular stellar system, restricting himself to Schwarzschild's exponential ellipsoidal distribution function. Jeans's theorem enables us to shorten considerably this work and to extend it to the general ellipsoidal distribution function. We shall now examine this point fully in the next section.

2. *Application of Jeans's Theorem to Globular Clusters.*—Let us take a Newtonian frame of reference at the centre of the cluster. Then since the gravitational force always passes through the centre, we shall have the three integrals of angular momentum, viz. :

$$I_1 = yw - zv\,; \qquad I_2 = zu - xw\,; \qquad I_3 = xv - yu, \tag{4}$$

and the integral of energy, viz. :

$$I = 2V - u^2 - v^2 - w^2 = 2V - c^2, \tag{5}$$

where V is the potential function.

And by Jeans's theorem the frequency function is given by

$$f = f(I\,;\ I_1, I_2, I_3). \tag{6}$$

We can now prove conversely that *a spherically symmetrical system is the only system having this form for f.*

For, that there should exist the integrals I_1, I_2, I_3 requires that the direction of the force of gravitation should always pass through the origin. This means that

$$\frac{\partial V}{\partial x}\Big/x = \frac{\partial V}{\partial y}\Big/y = \frac{\partial V}{\partial z}\Big/z.$$

Let each ratio be equal to $W \equiv W(x, y, z)$.

Then

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = W\{xdx + ydy + zdz\} = \frac{W}{2}d(r^2).$$

Now the right-hand side must be a perfect differential.

Therefore

$$W \equiv W(r)$$

and

$$V \equiv V(r),$$

which proves the result mentioned.

Jeans * has mentioned that I_1, I_2, I_3 can enter into the function f only through the form $(I_1^2 + I_2^2 + I_3^2)$, giving the reason that these integrals must, on account of the spherical symmetry, enter through a form symmetric in them and invariant with respect to a rotation of the axes of reference round the origin. But this point can be mathematically demonstrated in the following way :—

Let us write the fundamental equation in the spherical polar co-ordinates

* *Astronomy and Cosmogony* (1928), p. 357.

(r, θ, ϕ). Let the corresponding linear velocities be (R, Θ, Φ), and the corresponding linear accelerations (α, β, γ). Then

$$\left.\begin{aligned} R &= \dot{r} \\ \Theta &= r\dot{\theta} \\ \Phi &= r\sin\theta\,.\,\dot{\phi} \end{aligned}\right\} \qquad \left\{\begin{aligned} \alpha &= \dot{R} - \frac{\Theta^2+\Phi^2}{r} = \frac{\partial V}{\partial r}, \\ \beta &= \dot{\Theta} + \frac{R\Theta}{r} - \frac{\cot\theta}{r}\Phi^2 = \frac{\partial V}{r\partial\theta}, \\ \gamma &= \dot{\Phi} + \frac{R\Phi}{r} + \frac{\cot\theta}{r}\,.\,\Theta\Phi = \frac{\partial V}{r\sin\theta\partial\phi}, \end{aligned}\right\} \tag{7}$$

and

$$0 = \frac{Df}{Dt} \equiv \left(\frac{dr}{dt}\cdot\frac{\partial}{\partial r} + \frac{d\theta}{dt}\cdot\frac{\partial}{\partial\theta} + \frac{d\phi}{dt}\cdot\frac{\partial}{\partial\phi} + \frac{dR}{dt}\cdot\frac{\partial}{\partial R} + \frac{d\Theta}{dt}\cdot\frac{\partial}{\partial\Theta} + \frac{d\Phi}{dt}\cdot\frac{\partial}{\partial\Phi}\right)f. \tag{8}$$

(7) and (8) give

$$R\frac{\partial f}{\partial r} + \frac{\Theta}{r}\cdot\frac{\partial f}{\partial\theta} + \frac{\Phi}{r\sin\theta}\cdot\frac{\partial f}{\partial\phi} + \left(\frac{\partial V}{\partial r} + \frac{\Theta^2+\Phi^2}{r}\right)\frac{\partial f}{\partial R} + \left(\frac{\partial V}{r\partial\theta} - \frac{R\Theta}{r} + \frac{\cot\theta}{r}\,.\,\Phi^2\right)\frac{\partial f}{\partial\Theta}$$
$$+\left(\frac{\partial V}{r\sin\theta\partial\phi} - \frac{R\Phi}{r} - \frac{\cot\theta}{r}\,.\,\Theta\Phi\right)\frac{\partial f}{\partial\Phi} = 0.$$

For spherical symmetry we must have

$$\frac{\partial f}{\partial\theta} = \frac{\partial f}{\partial\phi} = \frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial\phi} = 0.$$

Thus

$$R\frac{\partial f}{dr} + \left(\frac{\partial V}{\partial r} + \frac{\Theta^2+\Phi^2}{r}\right)\frac{\partial f}{\partial R} + \left(-\frac{R\Theta}{r} + \frac{\cot\theta}{r}\,.\,\Phi^2\right)\frac{\partial f}{\partial\Theta} + \left(-\frac{R\Phi}{r} - \frac{\cot\theta}{r}\,.\,\Theta\Phi\right)\frac{\partial f}{\partial\Phi} = 0.$$

And since f cannot involve θ we must have separately

$$\left.\begin{aligned} &R\frac{\partial f}{\partial r} + \left(\frac{\partial V}{\partial r} + Q^2/r\right)\frac{\partial f}{\partial R} - \frac{R\Theta}{r}\cdot\frac{\partial f}{\partial\Theta} - \frac{R\Phi}{r}\cdot\frac{\partial f}{\partial\Phi} = 0, \\ \text{and} \qquad &\Phi^2\frac{\partial f}{\partial\Theta} - \Theta\Phi\frac{\partial f}{\partial\Phi} = 0, \end{aligned}\right\} \tag{9}$$

where

$$Q^2 \equiv \Theta^2 + \Phi^2.$$

The second equation of (9) shows that Θ and Φ enter into f only through $\Theta^2+\Phi^2$, *i.e.* Q^2, which is the same as $\dfrac{I_1^2+I_2^2+I_3^2}{r^2}$. The first equation of (9) now gives

$$f \equiv f(r, R, Q)$$

and

$$R\frac{\partial f}{\partial r} + \left(Q^2/r + \frac{\partial V}{\partial r}\right)\frac{\partial f}{\partial R} - \frac{RQ}{r}\cdot\frac{\partial f}{\partial Q} = 0.$$

The Lagrangian subsidiary equations are

$$\frac{dr}{R}=\frac{dR}{Q^2/r+\dfrac{\partial V}{\partial r}}=\frac{dQ}{-RQ/r},$$

whence

$$f=f(2V-R^2-Q^2,\ rQ)$$

is the general integral, and

$$R^2+Q^2=u^2+v^2+w^2=c^2.$$

This form could evidently have been written down directly from Jeans's theorem since $(2V-c^2)$ is the energy integral and rQ is the constant total angular momentum round the centre.

Eddington has obtained (in suitable units) for a globular cluster

$$\left.\begin{aligned}\sigma&=\frac{1}{4\pi}\cdot\frac{e^{2V}}{1+r^2}\\ \frac{d^2V}{du^2}&+\frac{e^{2V}}{u^2(1+u^2)}=0,\end{aligned}\right\}$$

or

$$\left.z\frac{d^2z}{du^2}-\left(\frac{dz}{du}\right)^2=\frac{1}{u^2(1+u^2)},\right\}\qquad(10)$$

where σ is the star-density,

$$u\equiv\frac{1}{r},\qquad z\equiv e^{-V}.$$

And

$$4\pi\sigma=\frac{1}{z^2(1+r^2)},$$

in which Schwarzschild's exponential ellipsoidal distribution function was assumed.

We shall now give the corresponding general results for the case of the general ellipsoidal distribution function. This is such that it is any function of a quadratic expression in the velocities. Thus the most general ellipsoidal distribution that we can have so as to conform to the form $f(2V-R^2-Q^2,\ rQ)$ is represented by

$$\psi(2V-R^2-Q^2+2k_1rQ-k_2r^2Q^2)\,.\,dR\,.\,2\pi QdQ,$$

where k_1 and k_2 are constants. The element $2\pi QdQ$ is implied by Q being a two-dimensional component of velocity.

Thus

$$f\equiv\psi\{2V-R^2-(1+k_2r^2)Q^2+2k_1rQ\}.\qquad(11)$$

At first sight (11) appears to give an ellipsoidal distribution. But we must remember that Q is a two-dimensional component of velocity and the equi-frequency velocity-surfaces

$$2V-R^2-(1+k_2r^2)Q^2+2k_1rQ=\text{constant}$$

are not ellipsoids in the (R, Θ, Φ)-space. For, on rationalizing the above equation in (R, Θ, Φ), we get an equation of the fourth degree in (R, Θ, Φ). The conclusion is that k_1 must be zero in order to obtain an ellipsoidal distribution. Thus we have

$$\left.\begin{aligned} f &= \psi(2V - R^2 - \overline{1 + k_2 r^2} Q^2), \\ &= \psi(2V - c_1{}^2), \\ c_1{}^2 &\equiv R^2 + (1 + k_2 r^2) Q^2 = R^2 + Q_1{}^2 \\ Q_1{}^2 &= (1 + k_2 r^2) Q^2. \end{aligned}\right\} \quad (12)$$

where

and

Let μ = mass of each star and G be the constant of gravitation, and let ν be the star-density. Let us drop the suffix 2 of k_2 as no longer necessary.

$$\left.\begin{aligned} \nu &= \int_{-\infty}^{+\infty}\int_0^{\infty} \psi(2V - c_1{}^2) 2\pi Q dQ dR, \\ \nu &= \frac{4\pi}{1 + kr^2}\int_0^{\infty} \psi(2V - c_1{}^2) c_1{}^2 dc_1. \end{aligned}\right\} \quad (13)$$

i.e.

And Poisson's equation gives

$$u^4 \frac{d^2V}{du^2} = -\frac{16\pi^2 G\mu}{1 + kr^2}\int_0^{\infty} \psi(2V - c_1{}^2) c_1{}^2 dc_1 \quad (14)^*$$

where $u \equiv \frac{1}{r}$.

Equations (13) and (14) are the generalisations of Eddington's formulæ (10). Eddington has obtained a solution in series of (14) for the particular case $\psi(x) = e^x$.

If a bar denote the mean value, the following are easily obtained :—

$$\left.\begin{aligned} \nu\overline{R^2} &= \frac{4\pi}{3(1 + kr^2)}\int_0^{\infty} \psi(2V - c_1{}^2) c_1{}^4 dc_1, \\ \nu\overline{Q^2} &= \frac{8\pi}{3(1 + kr^2)^2}\int_0^{\infty} \psi(2V - c_1{}^2) c_1{}^4 dc_1, \\ \nu\overline{\Theta^2} &= \nu\overline{\Phi^2} = \frac{4\pi}{3(1 + kr^2)^2}\int_0^{\infty} \psi(2V - c_1{}^2) c_1{}^4 dc_1. \end{aligned}\right\} \quad (15)$$

Thus

$$\nu\overline{R^2} : \nu\overline{\Theta^2} : \nu\overline{\Phi^2} : \nu\overline{Q^2} = 1 : \frac{1}{1 + kr^2} : \frac{1}{1 + kr^2} : \frac{2}{1 + kr^2}.$$

If k is positive, there will be radial star-streaming with the ratio of the axes of the velocity-ellipsoid equal to $\sqrt{1 + kr^2}$. If, however, k is negative, there will be star-streaming at right angles to the radius-vector. At the

* Strictly speaking, the upper limit of integration should be subject to the condition that the total velocity c does not exceed the "velocity of escape" $\sqrt{2V}$.

centre $r=0$ and $\nu\overline{R^2}=\nu\overline{\Theta^2}=\nu\overline{\Phi^2}$ and the distribution is spherical. Eddington's formulæ (10) corresponding to the exponential ellipsoidal distribution of Schwarzschild is obtained by taking $\psi(x)\equiv e^x$. The equations (13), (14) and (15) become, in this case,

$$\left.\begin{aligned} \nu &= \frac{\pi\sqrt{\pi}}{1+kr^2}e^{2V}, \\ u^4\frac{d^2V}{du^2} &= -\frac{4\pi^2\sqrt{\pi}}{1+kr^2}G\mu e^{2V}, \\ \nu\overline{R^2} &= \frac{\pi\sqrt{\pi}}{2(1+kr^2)}e^{2V}, \\ \nu\overline{Q^2} &= \frac{\pi\sqrt{\pi}}{(1+kr^2)^2}e^{2V}, \\ \nu\overline{\Theta^2} &= \nu\overline{\Phi^2} = \frac{\pi\sqrt{\pi}}{2(1+kr^2)^2}e^{2V}, \end{aligned}\right\} \tag{16}$$

which can be reduced to (10) by a suitable choice of units. Eddington's method * of deriving these was, of course, quite different, being independent of Jeans's theorem.

It is evident that $k=0$ in the general ellipsoidal distribution $\psi(2V-c_1^2)$ would give a spherical velocity-distribution of the general type. In particular if $\psi(x)=e^x$, the spherical distribution will be the well-known Maxwellian corresponding to statistical equilibrium. Jeans's † demonstration of Plummer's density law for a globular cluster corresponds to the particular case $\psi(x)=x^{7/2}$, $k=0$.

3. Formulæ corresponding to (13), (14) and (15) for the case of a rotating galaxy can be given as follows. Using cylindrical co-ordinates (ϖ, θ, z) let the corresponding velocities be (Π, Θ, Z). The first integrals of motion are

$$\left.\begin{aligned} I_1 &= \Pi^2+\Theta^2+Z^2-2V, \\ I_2 &= \varpi\Theta. \end{aligned}\right\} \tag{17}$$

For a general ellipsoidal distribution, f must be a function of a quadratic expression of Π, Θ, Z. Such an expression can be formed out of I_1 and I_2 only in the following most general way, viz.

$$\alpha(I_1+2k_1I_2+k_2I_2^2),$$

where α, k_1, k_2 are constants. The most general ellipsoidal distribution consistent with uniaxial symmetry is thus

$$f(I_1, I_2)=\phi(I_1+2k_1I_2+k_2I_2^2). \tag{18}$$

* Eddington, "Dynamics of a Globular Stellar System (II)," *M.N.*, **75**, 366, 1914.

† J. H. Jeans, "On the Law of Distribution in Star-clusters," *M.N.*, **76**, 567, 1916.

Now

$$I_1+2k_1I_2+k_2I_2^2=\Pi^2+\lambda^2(\Theta-\Theta_0)^2+Z^2-2V_1,$$

where

$$\left.\begin{aligned}\lambda^2&=1+k_2\varpi^2,\\ \Theta_0&=-\frac{k_1\varpi}{1+k_2\varpi^2},\\ 2V_1&=2V+\frac{k_1^2\varpi^2}{1+k_2\varpi^2},\qquad V=V(\varpi^2,z),\end{aligned}\right\}\tag{19}$$

and

$$f=\phi(\Pi^2+\lambda^2\,.\,\overline{\Theta-\Theta_0}^2+Z^2-2V_1)=\phi(c_1^2-2V_1),$$

where

$$c_1^2=\Pi^2+\Theta_1^2+Z^2,$$

and

$$\Theta_1=\lambda(\Theta-\Theta_0).$$

Now

$$\left.\begin{aligned}\nu&=\iiint_{-\infty}^{\infty}f\,d\Pi\,d\Theta\,dZ=\frac{4\pi}{\lambda}\int_0^{\infty}\phi(c_1^2-2V_1)c_1^2dc_1,\\ &\text{and Poisson's equation gives}\\ \frac{\partial^2V}{\partial\varpi^2}&+\frac{1}{\varpi}\frac{\partial V}{\partial\varpi}+\frac{\partial^2V}{\partial z^2}=-\frac{16\pi^2G\mu}{\lambda}\int_0^{\infty}\phi(c_1^2-2V_1)c_1^2dc_1.\end{aligned}\right\}\tag{20}$$

Write

$$p=\nu\overline{\Pi^2},\qquad q=\nu\overline{\Theta^2},\qquad q'=\nu\overline{(\Theta-\Theta_0)^2}.$$

Then the following can easily be obtained :—

$$\left.\begin{aligned}p&=\frac{4\pi}{3\lambda}\int_0^{\infty}\phi(c_1^2-2V_1)c_1^4dc_1,\\ q&=\nu\Theta_0^2+\frac{4\pi}{3\lambda^3}\int_0^{\infty}\phi(c_1^2-2V_1)c_1^4dc_1=\nu\Theta_0^2+q'=\nu\Theta_0^2+p/\lambda^2,\\ q'&=\frac{4\pi}{3\lambda^3}\int_0^{\infty}\phi(c_1^2-2V_1)c_1^4dc_1=p/\lambda^2.\end{aligned}\right\}\tag{21}$$

It is evident Θ_0 is the galactic rotation and λ is the ratio of the axes of the velocity ellipsoid. For radial star-streaming λ must be greater than unity. The formulæ

$$\left.\begin{aligned}q&=\nu\Theta_0^2+p/\lambda^2,\\ q'&=p/\lambda^2,\end{aligned}\right\}\tag{22}$$

are thus independent of the distribution of velocities so long as it is ellipsoidal.

It has been shown by Lindblad and others that

$$\lambda^{-1}=\sqrt{\left(1+\frac{A}{\omega}\right)},$$

where A is Oort's constant, and ω is the angular velocity, taken with its sign, of galactic rotation. This can also be derived as follows :—

From (19)

$$\Theta_0\varpi = -\frac{k_1\varpi^2}{1+k_2\varpi^2}.$$

Differentiating logarithmically with respect to ϖ,

$$\frac{1}{\Theta_0}\cdot\frac{\partial\Theta_0}{\partial\varpi}+\frac{1}{\varpi}=\frac{2}{\varpi(1+k_2\varpi^2)}=\frac{2}{\varpi\lambda^2}.$$

Also

$$A=\tfrac{1}{2}\left(\frac{\partial\Theta_0}{\partial\varpi}-\frac{\Theta_0}{\varpi}\right) \quad \text{and} \quad \omega=\frac{\Theta_0}{\varpi}.$$

Thus

$$\lambda^{-1}=\sqrt{\left(1+\frac{A}{\omega}\right)}.$$

Thus λ will be greater than unity, if A/ω is negative; that is if A is positive, since ω has been accepted to be negative. Again

$$2A=\varpi\frac{\partial}{\partial\varpi}\left(\frac{\Theta_0}{\varpi}\right)=\frac{\varpi}{2\omega}\frac{\partial}{\partial\varpi}(\omega^2).$$

Thus A will be positive if ω^2 decreases with increasing ϖ, *i.e.* if the angular velocity decreases numerically with increasing ϖ.

Formulæ (20) and (21) in the particular case of Schwarzschild's exponential ellipsoidal law are obtained by putting $\phi(x)=e^{-\alpha x}$, where α is a constant.

$\alpha > 0$ for obvious reasons. We have

$$\left.\begin{aligned}
\nu &= \left(\frac{\pi}{\alpha}\right)^{3/2}\cdot\frac{1}{\lambda}\cdot e^{2\alpha V_1},\\
\nu\overline{\Pi^2} &= p = \left(\frac{\pi}{\alpha}\right)^{3/2}\frac{1}{2\lambda\alpha}e^{2\alpha V_1},\\
\nu\overline{\Theta^2} &= q = \nu\Theta_0{}^2+p/\lambda^2 = \nu\Theta_0{}^2+q' = \nu\Theta_0{}^2+\left(\frac{\pi}{\alpha}\right)^{3/2}\frac{1}{2\lambda^3\alpha}e^{2\alpha V_1},\\
\nu\overline{(\Theta-\Theta_0)^2} &= q' = p/\lambda^2 = \left(\frac{\pi}{\alpha}\right)^{3/2}\frac{1}{2\lambda^3\alpha}\cdot e^{2\alpha V_1},
\end{aligned}\right\} \tag{23}$$

which further reduce to

$$\left.\begin{aligned}
\nu &= \left(\frac{\pi}{\alpha}\right)^{3}\cdot\frac{1}{\lambda}\cdot e^{2\alpha V_1},\\
\overline{\Pi^2} &= \frac{1}{2\alpha},\\
\overline{\Theta^2} &= \Theta_0{}^2+\frac{1}{2\alpha\lambda^2} = \overline{\Theta_0}^2+\overline{(\Theta-\Theta_0)^2},\\
\overline{(\Theta-\Theta_0)^2} &= \frac{1}{2\alpha\lambda^2}.
\end{aligned}\right\} \tag{24}$$

We can also write

$$f = \mathrm{Exp}\left[-\left\{ \frac{\Pi^2 + Z^2}{2a^2} + \frac{(\Theta - \Theta_0)^2}{2b^2} - \frac{2V_1}{2a^2} \right\} \right], \tag{25}$$

where

$$\left.\begin{aligned} &\alpha = \frac{1}{2a^2}, \qquad \alpha\lambda^2 = \frac{1}{2b^2}, \\ &\nu = \frac{1}{\lambda}(2\pi a^2)^{3/2} e^{V_1/a^2}, \\ &\overline{\Pi^2} = a^2, \\ &\overline{\Theta^2} = \Theta_0{}^2 + \overline{(\Theta - \Theta_0)^2} = \Theta_0{}^2 + a^2/\lambda^2 = \Theta_0{}^2 + b^2, \\ &\overline{(\Theta - \Theta_0)^2} = a^2/\lambda^2 = b^2. \end{aligned}\right\} \tag{26}$$

a, b, a are the semi-axes of the velocity-ellipsoid.

Sidney Sussex College,
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NOTE ON E. A. MILNE'S PAPER "STELLAR KINEMATICS AND THE *K*-EFFECT."

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The nature of the radial and transverse stellar motions, in the case of the most general systematic motion of the stellar system with specialisation for the two-dimensional case in any plane and in particular for a plane of symmetry like the galactic plane of our galaxy, has been studied by E. A. Milne.* The two-dimensional case had previously been treated by K. Ogrodnikoff.†

The paper discusses among other things the *K*-term, the "local solar motion," and the fact that, irrespective of the mode of systematic motion in the stars, the stellar motions relative to the Sun parallel to any plane corrected for the *K*-effect and the "local solar motion" consist of a term which is of the nature of a double wave in the angular co-ordinate in that plane. Previously Oort ‡ had shown how a rotation of the galaxy would lead to such a term involving galactic longitude in the radial and transverse stellar motions, and K. Ogrodnikoff † had extended this result to the general two-dimensional systematic motion. As the Oort-effect was used as a

* "Stellar Kinematics and the *K*-effect," *M.N.*, 95, 560, 1935.

† *Zeit. f. Astr.*, 4, 190, 1932. Milne was unaware of Ogrodnikoff's paper when he published his own. He has since made acknowledgment in a letter to the *Observatory* of 1935 August.

‡ *B.A.N.*, 3, 120, 1927.

reason for the conclusion that the galaxy is only rotating, Milne's and Ogrodnikoff's results might easily be misconceived as refuting the hypothesis of galactic rotation. Instead, Milne only generalises the Oort-effect, as he himself has said in the summary of his paper. However, some important facts emerge from these authors' papers which we will enunciate in a proper form below and show how they are evident from elementary considerations without going into the detail of Milne's paper.

It will be sufficient for this purpose to consider only our galaxy and the stellar motions in the galactic plane. Let (Π, Θ) denote velocities relative to galactic centre C along and at right angles to the radius-vector from C, and the suffix o denote systematic velocities. Let l denote galactic longitude and l_0 the galactic longitude of the centre C, referred to the galactic pole R.A. $= 18^h\ 40^m$, Dec. $= +28°$ and measured anti-clockwise. Let ρ_0 and σ_0 denote radial and transverse stellar motions relative to the Sun and ρ, σ the same as corrected for the K-effect and the "local solar motion." Let r denote distance from the Sun and (ϖ, θ) plane polar co-ordinates in the galactic plane referred to C, θ being measured in the same sense as l.

Then, for galactic rotation, $\Pi_0 = 0$ and

$$\left.\begin{aligned} \rho &= rA_0 \sin 2(l - l_0), \\ \sigma &= rA_0 \cos 2(l - l_0) + rB_0, \end{aligned}\right\} \qquad (1)^*$$

where

$$2A_0 = \frac{\partial \Theta_0}{\partial \varpi} - \frac{\Theta_0}{\varpi}, \qquad 2B_0 = \frac{\partial \Theta_0}{\partial \varpi} + \frac{\Theta_0}{\varpi}.$$

A_0 and B_0 are Oort's constants.

And for the general systematic motion, $\Pi_0 \neq 0$, $\Theta_0 \neq 0$ and

$$\left.\begin{aligned} \rho &= r(A + B \sin 2\lambda + C \cos 2\lambda), \\ \sigma &= r(A' - C \sin 2\lambda + B \cos 2\lambda), \end{aligned}\right\} \qquad (2)^\dagger$$

where

$$\lambda \equiv l - l_0,$$

$$2A = \frac{\Pi_0}{\varpi} + \frac{\partial \Pi_0}{\partial \varpi} + \frac{\partial \Theta_0}{\varpi \partial \theta},$$

$$2B = -\frac{\Theta_0}{\varpi} + \frac{\partial \Theta_0}{\partial \varpi} + \frac{\partial \Pi_0}{\varpi \partial \theta},$$

$$2C = -\frac{\Pi_0}{\varpi} + \frac{\partial \Pi_0}{\partial \varpi} - \frac{\partial \Theta_0}{\varpi \partial \theta},$$

$$2A' = \frac{\Theta_0}{\varpi} + \frac{\partial \Theta_0}{\partial \varpi} - \frac{\partial \Pi_0}{\varpi \partial \theta}.$$

By an analysis of stellar motions, Oort has found the values of $\overline{rA}$, $\overline{rB}$ and l_0 from the observational material for certain stars. The logicality of deriving the value of l_0 in this way by means of equations meant to test

* *B.A.N.*, 3, 120, 1927.

† E. A. Milne, *loc. cit.* See also a paper by the present author, "On the Direction of Star-streaming in the Galaxy," *M.N.*, 95, 655, 1935, for an independent derivation.

whether the galaxy is rotating is doubtful. Oort's determination of l_0 from stellar motions using the formulæ (1) is based on an *assumption* of a pure galactic rotation, as he has written his equations of condition in the form required by (1). Thus his results, by themselves, are not a proof of the rotation of the galaxy unless his value of l_0 comes out to be the actual one in nature, as determined by some independent means, such as Shapley's * value 327° derived from the positional distribution of the system of globular clusters. Whether Shapley's value is the one in nature is another matter. It is illogical, thus, to expect observations of stellar motions alone both to furnish the value of l_0 and to give proof of rotation. One of these will have to be assumed and the other can then be tested. This could mathematically be made clear in the following way. All search to obtain the double-wave terms in ρ_0 and σ_0 consists implicitly of analysing the observations in the following way :—

$$\rho_0 = \alpha_0 + (\alpha_1 \cos l + \beta_1 \sin l) + (\alpha_2 \cos 2l + \beta_2 \sin 2l) + \ \ldots \qquad (3)$$

The first term on the right-hand side is the "K-term," the second evidently is due to the "local solar motion" and the third is due to differential systematic motion. Thus

$$\rho = (\alpha_2 \cos 2l + \beta_2 \sin 2l) + \cdots \qquad (4)$$

The right-hand side can be put into the form $\overline{rA} \sin 2(l - l_0)$ where $\overline{rA}$ and l_0 could be expressed in terms of α_2 and β_2. Such a representation assumes pure galactic rotation and gives the value of l_0 on that assumption, and the degree to which l_0 agrees with the actual longitude of C would be a measure of the correctness of this assumption.

The right-hand side of (4) could as well be put into the form :

$$\overline{rB} \sin 2(l - l_0) + \overline{rC} \cos 2(l - l_0) \qquad (5)$$

as implied by (2) ; $\overline{rB}$, $\overline{rC}$ being expressed in terms of α_2, β_2, while l_0 is an already known quantity such as that in Shapley's determination. This representation has three constants in it, viz. $\overline{rB}$, $\overline{rC}$, l_0, while the right-hand side of (4) contains only two constants, α_2, β_2. Hence information independent of stellar motions has been imperative. We can say that the systematic motion is a rotation only when $\overline{rC}$ comes out to be negligible when the true value of l_0 is used in (5).

To sum up : *If we are making no assumption as to the mode of the systematic motion but are trying to find it out, l_0 will have to be known from sources other than stellar motions.*

This is the general theorem which Milne's paper proves, instead of contradicting the hypothesis of galactic rotation.

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* *Star Clusters*, p. 22, 1930.

STAR MODELS WITH HIGH CENTRAL CONCENTRATION OF DENSITY.

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(Communicated by Sir Arthur Eddington, F.R.S.)

Part I

1. This paper is intended as a contribution to the question—under what physical conditions can a star have a high degree of concentration of density towards the centre? For a perfect gas any distribution of density is a possible form of equilibrium provided we are free to assign the opacity arbitrarily at each point of the radius. Thus we can choose a model of the desired degree of concentration and calculate what law of variation of opacity would be required to make the model possible. In order to compare the law thus calculated with the standard law $\kappa = \kappa_0 \rho / T^{7/2}$ we express it as a variation of the coefficient κ along the radius. It is known that concentration of density towards the centre is produced by decreasing κ_0 inwards; but little information is available as to how large a diminution of opacity is required to bring about a given degree of concentration. The purpose of the following calculation is to enable us to form some idea of the quantitative relation, *i.e.* of the sensitivity of the concentration to changes in the inward variation of opacity.

We assume that the material obeys the perfect gas law. As it happens this assumption is justified for the model used in the calculations, the concentration not being high enough to introduce degeneracy. But the primary reason for treating a perfect gas model is that it represents the conditions most favourable to high concentration. Degeneracy corresponds to reduced compressibility of the material, and to obtain the same degree of central condensation it is necessary to have a stronger decrease of opacity inwards.

Various degrees of central concentration are most conveniently represented by polytropic models of appropriate index n. As an example of rather high concentration we take a model having the observed mass, radius and luminosity of the Sun and forming a polytrope of index $n = 4{\cdot}5$. The central density is then 6189 times the mean density or 8732 grs./c.c.

2. *Evaluation of Radiation Pressure and Temperature.*—Assuming then that the perfect gas equation of state is applicable to the material throughout the model we have, at any point,

$$\beta P = p_G = \left(\frac{\mathfrak{R}}{\mu}\right)\rho T, \tag{1}$$

$$(1-\beta)P = p_R = \tfrac{1}{3}aT^4, \tag{2}$$

$$P = p_G + p_R, \tag{3}$$

where P is the total pressure, p_G the gas pressure, p_R the radiation pressure,

ρ the density, T the temperature and μ the mean molecular weight at the point; $\mathfrak{R}$ is the gas constant per mol and a is Stefan's constant.

From (1) and (2) by eliminating T,

$$P=\left[\left(\frac{\mathfrak{R}}{\mu}\right)^4\frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3}, \tag{4}$$

and this is formally equivalent to the equation of state

$$P=K\rho^{1+\frac{1}{4\cdot5}} \tag{5}$$

for the model. Thus by equating (4) and (5)

$$\left[\left(\frac{\mathfrak{R}}{\mu}\right)^4\frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3}=K\rho^{1+2/9},$$

$$1-\beta=\left[\left(\frac{\mu}{\mathfrak{R}}\right)^4\frac{aK^3}{3}\rho^{-1/3}\right]\beta^4.$$

Using the polytropic relation $\rho/\rho_c=\theta^{4\cdot5}$, this reduces to

$$1-\beta=\left[\left(\frac{\mu}{\mathfrak{R}}\right)^4\frac{aK^3}{3}\rho_c^{-1/3}\right]\theta^{-3/2}\beta^4, \tag{6}$$

where θ is the Emden function corresponding to the polytropic index $n=4\cdot5$.

With the data for the Sun the constants of the model are evaluated : *

$$\left.\begin{aligned}K&=8\cdot433\ .\ 10^{14},\\ \rho_c&=8732\ \text{grs./c.c.},\\ P_c&=5\cdot533\ .\ 10^{19}\ \text{dynes/cm.}^2.\end{aligned}\right\} \tag{7}$$

(The suffix "c" denotes the values of the variables at the centre.) We assume that the molecular weight μ is constant and equal to unity. The quartic equation (6) for the ratio $\beta=p_G/P$ at any point in the model now becomes

$$1-\beta=\cdot001597\theta^{-3/2}\beta^4. \tag{8}$$

A number of points along the radius are selected, the coefficient of β^4 calculated for each point by substituting the value of θ corresponding to the point, and (8) is then solved for the positive real root β between 0 and 1.

From (2) and the polytropic relation

$$P=P_c\theta^{5\cdot5}$$

we have

$$T^4=\frac{3P_c}{a}(1-\beta)\theta^{5\cdot5},$$

which, with the appropriate values of the constants, reduces to

$$T^4=2\cdot172\ .\ 10^{34}(1-\beta)\theta^{5\cdot5}. \tag{9}$$

(8) determines $1-\beta$ and (9) determines T. In the table the points are

* See, *e.g.*, Eddington, *Internal Constitution of the Stars*, chap. iv.

labelled by the ratios r/R of their distance r from the centre, to the complete radius R of the star. Column 2 gives the fraction of the total mass contained within the sphere of radius r interior to the point. Columns 3 and 4 give respectively the values of $1-\beta$ and of T.

3. *The Coefficient of Opacity.*—From the equations of mechanical and radiative equilibrium the relation *

$$\eta\kappa = \frac{4\pi cGM}{L}\frac{dp_R}{dP} \tag{10}$$

can be deduced and applies at each point in the model. Here M is the total mass and L the luminosity of the configuration, c the velocity of light, and G the gravitational constant. κ is the opacity at the point, and if $L(r)$ is the rate of liberation of energy by the mass $M(r)$ within the sphere of radius r interior to the point, η is defined by

$$\eta = \frac{L(r)}{M(r)} \Big/ \frac{L}{M}. \tag{11}$$

For the model considered (10) becomes

$$\eta\kappa = 1{\cdot}318 \,.\, 10^4 \frac{dp_R}{dP}. \tag{12}$$

$\frac{dp_R}{dP}$ is given as a function of β by the relation †

$$\frac{dp_R}{dP} = \frac{(1-\beta)(44-36\beta)}{(44-33\beta)}. \tag{13}$$

From (12) and (13), with the calculated values of β we obtain the values of the function $\eta\kappa$ at the selected points (column 5 of the table).

The actual value of the opacity coefficient κ at any point in the model depends on the distribution of energy sources, *i.e.* on the value of η at the point. The simplest assumption is that the sources are uniformly distributed through the star, in which case $\eta \equiv 1$ and the values calculated for $\eta\kappa$ are then also the values of the coefficient of opacity κ (column 5). Column 6 gives the values of κ calculated on the assumption that the rate of generation of energy ϵ per grm. of the material is proportional to T^6; this gives a moderately high degree of concentration of the energy sources. (Actually considerably more points than are given in the table were used to calculate the value of the function η at the selected points in this case.) We add in the last column the theoretical values κ_s of the opacity coefficient derived on the basis of the physical formula (with Strömgren's value of the coefficient) ‡

$$\kappa_s = 10^{25{\cdot}59}\frac{\rho}{T^{3{\cdot}5}}. \tag{14}$$

The effect of concentrating the energy sources towards the centre is to reduce the calculated value of the opacity at every point, the relative change

* See Eddington, *loc. cit.*, p. 114.

† Formula (14) of Part II, with $n = 4{\cdot}5$.

‡ *Cf.* B. Strömgren, *Handbuch der Astrophysick*, Band VII, p. 124.

of opacity being larger the nearer the point is to the centre of the configuration (compare, for example, columns 5 and 6 of the table). In the limit at the boundary of the star where $\eta = 1$ for all distributions, the change of κ on varying the source distribution is zero. Thus the set of values of κ for the uniform distribution are in a sense maximum values of the opacity at the relevant points. We also note that, whereas the values κ_s found from the standard formula (column 7) decrease outwards along the radius, the calculated values (columns 5 and 6) exhibit a variation in the opposite sense. At the centre κ_s is larger than the calculated opacity. The two agree at some point in the star, the position of the point depending on the source distribution, but they diverge rapidly on both sides of the point.

TABLE

$\frac{r}{R}$	$\frac{M(r)}{M}$	$(1-\beta)$	$\cdot 10^{-6} \cdot T$	$\eta\kappa$	κ $(\epsilon \propto T^6)$	κ_s
·000	·000	·001586	76·6	15·24	...	86·24
·019	·036	·001728	72·4	16·61	3·13	81·45
·038	·193	·002142	62·6	20·60	5·99	70·60
·057	·400	·002807	52·1	27·02	11·9	59·00
·075	·576	·003707	43·2	35·73	21·0	49·06
·094	·704	·004829	36·0	46·58	33·0	41·18
·113	·792	·006160	30·5	59·57	47·3	35·38
·138	·868	·008263	24·9	78·17	67·9	28·84
·163	·914	·01074	20·7	104·6	95·6	24·29
·195	·948	·01436	16·8	140·5	133	20·07
·226	·968	·01858	14·0	182·8	177	16·96
·258	·980	·02345	11·7	232·2	228	14·59
·314	·991	·03391	8·82	340·0	337	11·52
·377	·996	·04837	6·58	493·1	491	9·210
·440	·998	·06622	4·98	687·3	687	7·584
·503	·999	·08800	3·78	931·6	933	6·397
·565	·999	·1143	2·85	1235	1235	5·501
·628	1·000	·1462	2·12	1614	1614	4·820

The ratio $\frac{\kappa}{\kappa_s}$, where κ is the calculated value, may be regarded as the extra factor which must be inserted in the standard formula (14) in order that our model should be physically possible. Part of this factor must be attributed to the effects of the guillotine correction, and, this alone being by no means sufficient, a remaining part must be attributed to the effect of other unknown sources of deviation. The factor κ/κ_s varies for the uniform distribution from ·177 at the centre to 335 at $r/R = ·628$, and for $\epsilon \propto T^6$, from about ·03 at the centre to 335 at $r/R = ·628$.

A comparison of κ_s with κ for the uniform distribution shows that, in the inner core containing about 90 per cent. of the total mass, the discrepancy between the two coefficients is not very large, whereas it becomes considerable

in the stellar envelope which contains the exterior 10 per cent. of the mass. Thus we have to postulate in the envelope, sources of opacity which contribute very much more to the total opacity than the sources implied in the formula (14). This applies also to the case $\epsilon \propto T^6$, and indeed, to a very large extent, to all distributions of the energy sources, in particular to those in which the rate of generation of energy does not increase outwards along the radius. (It was on the assumption of the latter, *i.e.* that η does not increase outwards, that the κ for the uniform distribution were referred to above as, in a sense, maximum values.)

An alternative procedure is to accept the values κ_s obtained from (14) for the opacity and to calculate from them and from the known values of $\eta\kappa$ the function η at the specified points. This leads to the result that η increases from the centre outwards, reaches a maximum and then again decreases, *i.e.* the rate of generation of energy by the stellar material increases outwards up to a definite point and then starts decreasing. We do not give the details of the investigation here as this behaviour of the function η is immediately deducible from the theorem of Part II of this paper.

From the theorem, the rate of generation of energy by the material will increase outwards up to the point at which the ratio $1-\beta$ of radiation pressure to total pressure is ·6772 (see (24) of Part II), before it starts decreasing. This point corresponds to $r/R \doteqdot ·98$, and here the mass enclosed in a sphere of radius r through the point is indistinguishable to within ·001 per cent. from the total mass of the star. Thus the rate of generation increases from the centre till far out in the stellar envelope. As remarked at the conclusion of Part II of this paper, no adequate physical reasons can as yet be advanced against an assumption that energy generation in a star increases outward in the core and even in the envelope. It does, however, appear very unlikely that the rate of generation will increase up to very nearly the outer boundary of the envelope, as we have found it to do for our model.

It thus appears that, although for the inner core the calculated results are not widely different from those deduced from equation (14), it becomes increasingly difficult to reconcile the two as we proceed outwards in the stellar envelope. This applies to both types of consideration used above, *i.e.* modification of the physical formula (14), and distribution of energy sources.

Part II

1. *Theorem.*—Corresponding to a distribution law

$$P = K\rho^{1+\frac{1}{n}} \tag{1}$$

and an opacity law

$$\kappa \propto \frac{\rho}{T^s} \tag{2}$$

for a perfect gas configuration, there exists a value α^* of the ratio α of radiation-pressure * to total pressure, depending only on the indices n

* We write $1-\beta=\alpha$.

and s, such that in any (perfect gas) configuration for which the distribution of density and opacity is specified by the relations (1) and (2), the rate of generation of energy by the stellar material will be a monotonic decreasing function of distance from the centre only at those points at which $\alpha \geqslant \alpha^*$. (We assume $\mu =$ constant.)

2. *Proof.*—With the notation of Part I of this paper, for a perfect gas

$$\beta P = p_G = \left(\frac{\mathfrak{R}}{\mu}\right)\rho T, \tag{3}$$

$$(1-\beta)P = p_R = \tfrac{1}{3}aT^4. \tag{4}$$

Eliminating T we obtain the distribution law

$$P = \left[\left(\frac{\mathfrak{R}}{\mu}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{\frac{1}{3}}\rho^{\frac{4}{3}}. \tag{5}$$

Since (1) and (5) are formally equivalent, by equating their right-hand sides

$$1-\beta = \left[\left(\frac{\mu}{\mathfrak{R}}\right)^4\frac{a}{3}K^3\rho^{\frac{3-n}{n}}\right]\beta^4.$$

Using * the polytropic relation $\rho/\rho_c = \theta^n$ where θ is the Emden function corresponding to the polytropic index n, this reduces to

$$1-\beta = C_1\theta^{3-n}\beta^4, \tag{6}$$

where

$$C_1 = \left[\left(\frac{\mu}{\mathfrak{R}}\right)^4\frac{a}{3}K^3\rho_c^{\frac{3-n}{n}}\right].$$

C_1 is a constant for the particular configuration † and is essentially positive.

From (4), by using the polytropic relation $\dfrac{P}{P_c} = \theta^{n+1}$, we have

$$T^4 = C_2^4(1-\beta)\theta^{n+1},$$

where

$$C_2^4 = \frac{3P_c}{a},$$

therefore

$$T = C_2(1-\beta)^{\frac{1}{4}}\theta^{\frac{n+1}{4}}. \tag{7}$$

From (6)

$$(1-\beta)^{\frac{1}{4}} = C_1^{\frac{1}{4}}\beta\theta^{\frac{3-n}{4}}.$$

Substitute in (7) and put $C_2C_1^{\frac{1}{4}} = C_3$. Then

$$T = C_3\beta\theta. \tag{8}$$

At this stage it is convenient to express P, ρ, T, etc. as functions of β as independent variable instead of r. This procedure, which depends on the application of (6), breaks down for $n = 3$, as (6) is then independent of θ.

* The suffix "c" denotes the value at the centre.

† We shall use the "C_i" to denote constants for the particular configuration.

From (6)

$$\theta = C_1^{-\frac{1}{3-n}}\left[\frac{1-\beta}{\beta^4}\right]^{\frac{1}{3-n}}, \tag{9}$$

and so for the polytrope of index n

$$\left.\begin{aligned} \rho &= A_1\theta^n = B_1\left[\frac{1-\beta}{\beta^4}\right]^{\frac{n}{3-n}}, \\ P &= A_2\theta^{n+1} = B_2\left[\frac{1-\beta}{\beta^4}\right]^{\frac{n+1}{3-n}}, \\ T &= C_3\beta\theta = B_3\beta\left[\frac{1-\beta}{\beta^4}\right]^{\frac{1}{3-n}}, \end{aligned}\right\} \tag{10}$$

where the A_i and B_i are constants for the particular configuration.

3. *The derivative* $\frac{dp_R}{dP}$.—By differentiating the relation $p_R = (1-\beta)P$ with respect to P, we obtain

$$\frac{dp_R}{dP} = (1-\beta) - P\frac{d\beta}{dP}. \tag{11}$$

Also from (5)

$$\log P = \tfrac{1}{3}\log\left[\left(\frac{\Re}{\mu}\right)^4\frac{3}{a}\right] + \tfrac{1}{3}\log(1-\beta) - \tfrac{4}{3}\log\beta + \tfrac{4}{3}\log\rho.$$

Differentiate with respect to P. Then

$$\frac{1}{P} = -\frac{1}{3(1-\beta)}\frac{d\beta}{dP} - \frac{4}{3\beta}\frac{d\beta}{dP} + \frac{4}{3\rho}\frac{d\rho}{dP},$$

and so *

$$P\frac{d\beta}{dP} = \left\{\frac{4}{3}\frac{P}{\rho}\frac{d\rho}{dP} - 1\right\}\frac{3\beta(1-\beta)}{4-3\beta}. \tag{12}$$

By logarithmic differentiation of (1)

$$\frac{P}{\rho}\frac{d\rho}{dP} = \frac{n}{n+1},$$

and substituting this in (12)

$$P\frac{d\beta}{dP} = \frac{(n-3)\beta(1-\beta)}{(n+1)(4-3\beta)}. \tag{13}$$

Finally from (11) and (13) we obtain

$$\frac{dp_R}{dP} = \frac{4(1-\beta)(n+1-n\beta)}{(n+1)(4-3\beta)}. \tag{14}$$

The opacity law (2), *i.e.* $\kappa \propto \rho/T^s$ can, by (8) and the relation $\rho = \rho_c\theta^n$, be written in the form

$$\kappa \propto \frac{\theta^{n-s}}{\beta^s}.$$

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* *Cf.* Chandrasekhar, *Stellar Configurations with Degenerate Cores*, *M.N.*, 95, p. 685, equation (38).

This when expressed entirely in terms of β by applying (9) gives the opacity in the form

$$\kappa = C_4 \beta^{-s} \left[\frac{1-\beta}{\beta^4}\right]^{\frac{n-s}{3-n}}, \tag{15}$$

where C_4 is a constant for the particular configuration.

With the symbols defined as in Part I of this paper, the opacity κ and the average η for the rate of liberation of energy within a sphere of radius r,

$$\eta = \frac{L(r)/M(r)}{L/M},$$

are, at any point, connected by the equation *

$$\eta\kappa = \frac{4\pi cGM}{L}\frac{dp_R}{dP}. \tag{16}$$

$\eta\kappa$ can by (14) be expressed as a function of β,

$$\eta\kappa = \frac{4\pi cGM}{L}\cdot\frac{4}{n+1}\cdot\frac{(1-\beta)(n+1-n\beta)}{(4-3\beta)}. \tag{17}$$

By division of (17) with (15) we obtain an expression for η at any point as a function of β at the point.

$$\eta = C_5 \cdot \frac{(1-\beta)(n+1-n\beta)}{4-3\beta}\beta^s\left[\frac{\beta^4}{1-\beta}\right]^{\frac{n-s}{3-n}}, \tag{18}$$

where C_5 is a constant for the particular configuration.

$$C_5 = \frac{4\pi cGM}{L}\cdot\frac{4}{n+1}\cdot\frac{1}{C_4}.$$

It is convenient for the subsequent work to use $\alpha = 1-\beta$ as the independent variable instead of β. (18) then becomes

$$\eta = C_5 \alpha^{\frac{3-2n+s}{3-n}}(1-\alpha)^{\frac{4n-ns-s}{3-n}}\left[\frac{(1+n\alpha)}{(1+3\alpha)}\right]. \tag{19}$$

Differentiate $\log\eta$ with respect to α and simplify the resulting expression and we have finally

$$\frac{d(\log\eta)}{d\alpha} = \frac{F(n, s, \alpha)}{D(n, \alpha)}, \tag{20}$$

where

$$F(n, s, \alpha) = (2n-s-3) + (3n^2-n-2sn-3s+3)\alpha + (7n^2-sn^2-6sn+6n)\alpha^2 + (6n^2-3sn^2+9n)\alpha^3 \tag{21}$$

and

$$D(n, \alpha) = (n-3)\alpha(1-\alpha)(1+n\alpha)(1+3\alpha). \tag{22}$$

4. The sign of the right-hand side of (20) is, for any given opacity law, *i.e.* a given s, to be considered for $0 \leqslant \alpha \leqslant 1$ and for $n \leqslant 5$. We shall confine

* Compare Part I, equation (10).

our discussion to the case where the opacity is given by the standard law $\kappa \propto \rho/T^{3\cdot5}$. Equation (20) then becomes

$$\frac{d(\log \eta)}{d\alpha} = \frac{(4n-13)+(6n^2-16n-15)\alpha+(7n^2-30n)\alpha^2+(18n-9n^2)\alpha^3}{2(n-3)\alpha(1-\alpha)(1+n\alpha)(1+3\alpha)} = \frac{F(n,\ 3\cdot5,\ \alpha)}{2D(n,\ \alpha)}. \tag{23}$$

We shall treat separately the two cases (i) $n < 3$ and (ii) $n > 3$.

(1) $n < 3$.—The denominator $D(n,\ \alpha)$ of (23) is negative and, as is easily shown, the numerator $F(n,\ 3\cdot5,\ \alpha)$ is also negative, for all values of α in the range $0 \leqslant \alpha \leqslant 1$. Thus here

$$\frac{d(\log \eta)}{d\alpha} > 0 \qquad 0 \leqslant \alpha \leqslant 1,$$

i.e. $\log \eta$ and hence η, is a monotonic increasing function of α in the whole interval $0 \leqslant \alpha \leqslant 1$.

In equation (6), *i.e.*

$$1-\beta_1 = C_1\theta^{3-n}\beta^4, \tag{6}$$

since θ decreases from 1 at the centre to zero at the boundary, the coefficient $C_1\theta^{3-n}$ of β^4 also decreases from C_1 at the centre to zero at the boundary. (6) shows immediately that $\alpha = 1-\beta$ decreases from some fractional value at the centre to zero at the boundary. Since η is a monotonic increasing function of α throughout the range of permissible values of α, η will be a monotonic decreasing function of distance from the centre in all perfect gas configurations for which $n < 3$, and $s = 3\cdot5$. Thus the α^* of the theorem can in this case effectively be put equal to zero.

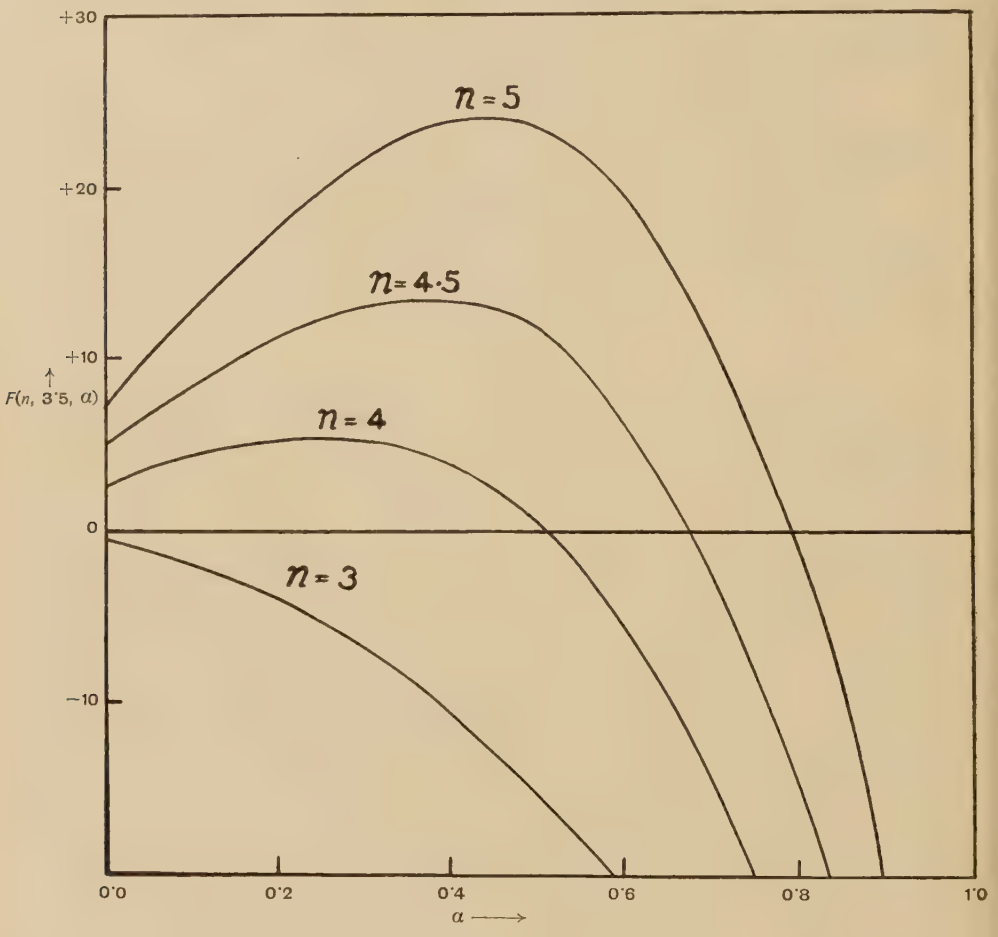
(ii) $n > 3$.—The denominator $D(n,\ \alpha)$ of (23) is in this case positive and so the sign of $\dfrac{d(\log \eta)}{d\alpha}$ is the same as that of the numerator $F(n,\ 3\cdot5,\ \alpha)$. In the figure this numerator $F(n,\ 3\cdot5,\ \alpha)$ is plotted as ordinate with α as abscissæ.

For $n \leqslant 3\cdot25$, $F(n,\ 3\cdot5,\ \alpha)$ is negative for all α in $0 \leqslant \alpha \leqslant 1$. (In this case we may again put $\alpha^* = 0$.) For $n > 3\cdot25$, $F(n,\ 3\cdot5,\ \alpha)$ is positive for values of α between zero and some value α^*, depending on n, and is negative in the range $\alpha^* < \alpha \leqslant 1$. The same variation of sign then also applies to the derivative $\dfrac{d(\log \eta)}{d\alpha}$.

In (6), with $n > 3$, θ decreases from 1 at the centre to zero at the boundary, and thus the coefficient $C_1\theta^{3-n}$ of β^4 increases from C_1 at the centre, tending to infinity at the boundary. Therefore $\alpha = 1-\beta$ will increase from some fractional value at the centre to 1 at the boundary.

Now η is an increasing function of α for $0 \leqslant \alpha < \alpha^*$, and a decreasing function for $\alpha^* < \alpha < 1$, *i.e.* η is an increasing function of distance from the centre up to the point where $\alpha = \alpha^*$, and after this point becomes a monotonic decreasing function of the distance, tending to the value unity at the boundary.

This completes the proof of the theorem in the special case $s = 3{\cdot}5$. The procedure for other values of s is exactly analogous.



$F(n, 3{\cdot}5, \alpha)$ is plotted as a function of α for $n = 3, 4, 4{\cdot}5, 5$. α^* is given by the root between 0 and 1. If there is no root between 0 and 1, $\alpha^* = 0$.

For $s = 3{\cdot}5$ we have, for example, by solving the cubic equation

$$F(n, 3{\cdot}5, \alpha^*) = 0,$$

for the positive real root between 0 and 1 :

$$\left.\begin{array}{ll} n \leqslant 3{\cdot}25 & \alpha^* = 0{\cdot}0000, \\ n = 4 & \alpha^* = 0{\cdot}5108, \\ n = 4{\cdot}5 & \alpha^* = 0{\cdot}6772, \\ n = 5{\cdot}0 & \alpha^* = 0{\cdot}7867. \end{array}\right\} \qquad (24)$$

5. If we assume that the rate of generation of energy does not increase outwards in a star, *i.e.* that η does not increase outwards along the radius, then the theorem imposes a limitation upon the possible perfect gas configurations in which the distribution of density is specified by (1) and the opacity

by (2). This limitation would consist in the condition that the value of the ratio α of radiation pressure to total pressure at the centre of the configuration must not be less than α^*. We have, however, preferred to state the theorem in a form which is independent of any restrictions as to possible distributions of the energy sources.

No adequate physical reasons can as yet be advanced which would lead us to exclude an assumption to the effect that the rate of generation of energy ϵ increases outward along the radius. It does, however, appear probable that as we move outwards ϵ will, from some point well within the envelope, become, and will remain, a decreasing function of distance from the centre. Thus we may with tolerable certainty rule out as impossible those configurations in which ϵ starts decreasing only at a point very close to the outer boundary of the envelope. Thus, for example, for a star of ordinary mass the value of α at the centre is of the order of about ·01, and for $n > 3$ increases outwards. For such a perfect gas configuration specified by (1) and (2) with $4 \leqslant n \leqslant 5$ and $s = 3{\cdot}5$, the point at which $\alpha = \alpha^*$ is certainly far out in the envelope, and this would lead us to exclude such configurations as probably being physically impossible.

In conclusion it is a pleasure to record my thanks to Sir Arthur Eddington who suggested the problem, and to Dr. Chandrasekhar, for their helpful criticism and advice during the course of this work.

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1936 *June.*

THE THEORETICAL CAPTURE SPECTRUM OF HYDROGEN.

G. G. Cillié.

1. *Introduction.*—The discrete Balmer spectrum of hydrogen and the continuum at its head is observed in emission in planetary nebulæ, in novæ and in the solar chromosphere. An attempt has been made to account for the observed Balmer decrement in the spectra of planetary nebulæ on the basis of Zanstra's theory of nebular excitation.* The list of known Einstein transition coefficients for hydrogen has been considerably extended recently by Menzel and Pekeris.† With these it is possible to repeat the previous calculations, taking into account the states of hydrogen up to the fourteenth, instead of only up to the seventh as was done before. It is possible to extend the theoretical Balmer decrement up to H_{10}. In addition, the results show by comparison in how far we are justified in neglecting the upper states.

* *M.N.*, **92**, 820, 1932.
† *M.N.*, **96**, 77, 1935.

As before, the hydrogen spectrum is regarded as due to captures of free electrons by protons, line excitation from subordinate levels is neglected. Capture coefficients for the various levels as derived from Gaunt's absorption coefficients * are used. The relative numbers of hydrogen atoms per c.c. in state n, N_n, are determined for different values of the temperature of the free electrons, T_e. In every case N_n is found in terms of N_iN_e, where N_i and N_e are the numbers of protons and free electrons per c.c. respectively. The intensity of the continuous Balmer spectrum depends upon the product N_iN_e. Hence the intensities of the lines in the Balmer series are found in terms of the intensity of the continuum at the head of the series. These quantities are more sensitive to changes in the electron temperature than the Balmer decrement itself. The computed values can be compared with the observed ones for the chromospheric emission spectrum † and with Page's ‡ observed intensities of the Balmer spectrum in certain planetary nebulæ.

The relative numbers of electron captures on the various hydrogen levels for different electron temperatures can be derived from the known capture coefficients. The emission in Balmer lines near the limit arise from captures in the high states. Hence the variation with electron temperature in the intensities of the upper Balmer lines relative to the intensity in the Balmer continuum can be predicted.

2. *The Theoretical Balmer Decrement.*—The number of electrons captured by protons in state n of hydrogen per second per c.c. is given by §

$$N_iN_eC(n,\ T_e) = N_iN_e\frac{2^9\pi^5}{(6\pi)^{3/2}}\frac{\epsilon^{10}}{m^2c^3h^3}\left(\frac{m}{kT_e}\right)^{3/2}\frac{1}{n^3}e^{\chi_n/kT_e}\{-Ei(-\chi_n/kT_e)\}, \qquad (2.1)$$

where T_e is the electron temperature and χ_n the ionisation potential of the nth state.

The electron, after capture, cascades down until it reaches the lowest or normal state. The hydrogen is assumed to be completely transparent to the ensuing subordinate radiations. It is further assumed that the hydrogen is so opaque to Lyman line radiation that the number of transitions from the nth to the lowest level is exactly balanced by the number of excitations from the lowest to the nth level. ||

The net number of atoms per c.c. leaving state n spontaneously every second for lower states is

$$N_n\sum_{s=2}^{n-1}A_{n,\,s}, \qquad (2.2)$$

where $A_{n,\,s}$ is the usual Einstein coefficient of spontaneous transition.

* *Phil. Trans.*, **229** A, 200, 1929.

† Menzel and Cillié, in press.

‡ Page, "The Continuous Spectrum of Certain Planetary Nebulæ," *M.N.*, **96**, 604 (1936).

§ *M.N.*, **92**, 823, eq. (7), 1932, where we introduce $-Ei(-\chi)$ for $Ei(x)$.

|| These conditions were introduced for nebulæ by Zanstra, *Pub. D.A.O.*, **4**, 200, 1931.

The number of atoms arriving in state n per c.c. per second from higher states by spontaneous transition is

$$\sum_{s=n+1}^{14} N_s A_{s,n}, \tag{2.3}$$

where we neglect states above the fourteenth. If further we assume that stimulated transitions may be neglected, we find on combining (2.1), (2.2) and (2.3), for the equilibrium of atoms in the nth state, that

$$N_n \sum_{s=2}^{n-1} A_{n,s} - \sum_{s=n+1}^{14} N_s A_{s,n} = N_i N_e C(n, T_e). \tag{2.4}$$

Since in the present problem n ranges from 3 to 14, (2.4) gives rise, for any value of T_e, to twelve linear simultaneous equations involving the twelve unknowns $\frac{N_n}{N_i N_e}$. We solve the system of equations for five different values of the electron temperature, namely, 1000°, 5000°, 10,000°, 20,000° and 50,000°. The results are given in Table I. From Table I the theoretical Balmer decrement can be calculated, since the emission in ergs per c.c. per second in the line H_n is given by

$$E_n = N_n A_{n,2} h\nu_{n2}. \tag{2.5}$$

TABLE I

Values of $\frac{N_n}{N_i N_e} \times 10^{20}$

n	$T_e = 1000°$	5000°	10,000°	20,000°	50,000°
3	...	0·408	0·233	0·126	0·0515
4	1·88	·589	·327	·171	·0667
5	2·98	·889	·480	·244	·0919
6	4·52	1·287	·677	·336	·1232
7	6·54	1·781	·917	·447	·1600
8	9·04	2·359	1·193	·571	·2008
9	11·99	3·011	1·498	·707	·2448
10	15·37	3·727	1·830	·852	·2911

Choosing the intensity of $H\beta$ as 1·00, we give in Table II the intensities of the other members of the series up to H_{10}. The relative intensities of $H\alpha$, $H\beta$ and $H\gamma$ correspond closely with those given previously, indicating that the higher states taken into account in the present paper have little influence on the numbers of atoms in the lower states.

Comparison of the different columns in Table II confirms our previous conclusion, namely, that the theoretical Balmer decrement is relatively insensitive to the temperature of the electrons. As suggested before, the failure of the above theory to account for the observed variation in the

Balmer decrement in planetary nebulæ * may be due to an assumption inherent in the method, namely, that at any moment the substates of any level are equally occupied.

TABLE II

The Theoretical Balmer Decrement

H_n	$T_e=1000°$	5000°	10,000°	20,000°	50,000°
$H\alpha$	...	2·70	2·78	2·88	3·01
$H\beta$	1·00	1·00	1·00	1·00	1·00
$H\gamma$	·54	·51	·50	·48	·47
$H\delta$	·33	·30	·29	·27	·25
$H\epsilon$	·22	·193	·180	·168	·154
H_8	·158	·134	·120	·110	·099
H_9	·117	·094	·084	·076	·067
H_{10}	·088	·069	·061	·054	·047

3. *The Balmer Continuous Spectrum.*—In view of the results of the previous section we are unable to deduce the electron temperature from the observed Balmer decrement. It was suggested before that the electron temperature may be inferred from the intensity distribution in the Balmer continuum. The energy emitted per second per c.c. in a frequency range $d\nu$ of this spectrum is †

$$N_i N_e \frac{2^6\pi^5}{(6\pi)^{3/2}} \frac{\epsilon^{10}}{m^2c^3h^2}\left(\frac{m}{kT_e}\right)^{3/2} e^{\chi(2-h\nu)/kT_e} d\nu. \qquad (3.1)$$

Determination of T_e by means of this formula involves the measurement of intensities in the Balmer continuum in the far ultra-violet beyond λ 3647. In view of the difficulty of obtaining suitable standards in this region, it is desirable to find some other method by which to determine the electron temperature from the observed Balmer spectrum.

The emission per unit wave-number per c.c. per second at the head of the Balmer series is, by (3.1),

$$Ba_c = [\overline{23}\cdot902]\frac{N_i N_e}{T_e^{3/2}}, \qquad (3.2)$$

where we have introduced the values of the physical constants. From (2.5) and (3.2) we find that

$$E_n/Ba_c = [22\cdot098]\frac{N_n}{N_i N_e} A_{n,2} h\nu_{n2} T_e^{3/2}. \qquad (3.3)$$

This quantity represents the intensity of the Balmer line H_n, relative to the intensity per unit wave-number in the continuum at the limit of the series. The theoretical values of this ratio, computed from Table I by means of (3.3), are given in columns 4 to 8 in Table III. The first three columns of Table III give respectively n, H_n and the wave-length of the line.

* Plaskett, *H.C.O. Circular*, 335, 1928 ; Berman, *L.O.B.*, 430, 1930.

† *M.N.*, **92**, 827, eq. (15), 1932.

TABLE III

Calculated Values of $\log_{10} E_n/Ba_c$

n	H_n	λ	1000°	5000°	10,000°	20,000°	50,000°
3	$H\alpha$	6562·8	...	3·378	3·601	3·770	3·978
4	$H\beta$	4861·3	2·402	2·946	3·141	3·311	3·500
5	$H\gamma$	4340·5	2·130	2·654	2·837	2·995	3·168
6	$H\delta$	4101·8	1·921	2·424	2·596	2·744	2·906
7	$H\epsilon$	3970·1	1·750	2·232	2·396	2·535	2·686
8	H_8	3889·1	1·601	2·066	2·222	2·353	2·496
9	H_9	3835·4	1·470	1·918	2·067	2·191	2·229
10	H_{10}	3797·9	1·348	1·782	1·924	2·044	2·175

To determine E_n/Ba_c from observation involves in the case of $H\alpha$ a comparison of intensities in regions of the spectrum as well separated as λ 6563 and λ 3647. Over such a wide range it is difficult to attain great accuracy in the relative intensities. The same applies to a lesser extent to $H\beta$ and $H\gamma$. In addition, the above theory does not seem to be applicable to the first few states of hydrogen, in view of the failure to explain the observed variation in the Balmer decrement. The higher members of the series approach more and more closely to λ 3647, and the intercomparison between E_n and Ba_c may be made more readily.

Table IV gives some of the observed values of $\log_{10} E_n/Ba_c$. Columns 2 and 3 refer to simultaneous observations of the 1932 eclipse,* made with an ultra-violet and a grating spectrograph respectively. In the case of the latter instrument the spectrum faded out to the redward side of λ 3647. The intensity of the Balmer continuum at the series limit, Ba_c, as observed on the ultra-violet spectrum, is used in connection with the line intensities derived from both kinds of spectra.

TABLE IV

Observed Values of $\log_{10} E_n/Ba_c$

H_n	Chromosphere		N.G.C. 7662		N.G.C. 7027
	U.V. Film	Red Film	West Side	East Side	
$H\alpha$	...	4·28	...	...	5·47
$H\beta$	3·58	3·36	4·08	3·81	4·71
$H\gamma$	2·99	2·79	3·68	3·42	4·28
$H\delta$	2·65	2·54	3·34	3·09	3·79
$H\epsilon$	...	2·39	...	...	...
H_9	2·17	2·13	...	...	3·15
H_{10}	1·93	1·91	...	...	...

* Taken from Menzel and Cillié, *loc. cit.*, Tables 7 (*a*) and 7 (*c*).

Comparing these values with those given in Table III, we find that from $H\gamma$ to H_{10} they agree closely with the values computed for $T_e = 10{,}000°$. Measurement of the intensity distribution in the Balmer continuous spectrum of the chromosphere gave an electron temperature of this same order.* It should be added that although the measurements quoted in columns 2 and 3 of Table IV refer to a region of the chromosphere lying 670 km. above the observed base, the first few members of the Balmer series are affected by self-reversal, and on this account the first few entries in these columns should be slightly increased. Corrections for self-reversal, however, do not affect the correspondence between calculation and observation for the higher members of the series. The agreement suggests that the chromospheric hydrogen spectrum is largely due to capture.

The values of $\log_{10} E_n/Ba_c$ in columns 4, 5 and 6 of Table IV refer to Page's measurements of the hydrogen spectrum in certain planetary nebulæ.† From the intensity distribution in the Balmer continuous spectra of these objects, Page derives an electron temperature of the order of 2000°. Comparison between these values of $\log_{10} E_n/Ba_c$ and the calculated ones indicate that the values of E_n/Ba_c computed for a purely capture spectrum, without any reversal of the subordinate lines, are a hundred times smaller than the observed ones.

It has been pointed out by Ambarzumian ‡ that the density of Ly_α radiation in planetary nebulæ is considerable, and that the number of hydrogen atoms in the first excited state will be appreciable. In addition, the metastability of this level causes an accumulation of atoms in the state. § Hence it is possible that the Balmer spectra of planetary nebulæ are affected by self-reversal.‖

The considerable difference between the computed and observed values of E_n/Ba_c, if it is due to self-reversal, would mean that even the Balmer continuous spectrum suffers reversal in planetary nebulæ. In this case these objects have appreciable optical thicknesses in the lower members of the Balmer series.

Self-reversal affects the Balmer decrement in two ways. It increases the concentration of hydrogen atoms in the lower states compared with that in the higher ones. This renders the Balmer decrement of the emission of any c.c. more steep than that given in Table II. On the other hand, since the absorption coefficients for the Balmer lines decrease with the quantum number of the upper state, the emergent radiation originates in deeper layers of the nebulæ in the case of the higher members. This tends to decrease the Balmer decrement. It is possible that a combination of these two effects may account for part of the discrepancy between the Balmer decrement in Table II and the observed one.

* Menzel and Cillié, *loc. cit.*
† Page, *loc. cit.*, Tables IV, V, VI and VII.
‡ *Poulkovo Bulletin*, **13**, No. 3, 1933.
§ Stoy, Ph.D. thesis, Cambridge, 1936.
‖ First suggested by Chandrasekhar. See Page, *loc. cit.*

4. *Members of the Balmer Series near the Limit.*—By (2.1) the number of captures on level n is proportional to

$$\frac{N_i N_e}{T_e^{3/2}} \frac{1}{n^3} e^{\chi_n/kT_e}\{-Ei(-\chi_n/kT_e)\}. \tag{4.1}$$

When n is large, and for the range of T_e in which we are interested, χ_n/kT_e is small. Hence for n large we may take

$$e^{\chi_n/kT_e} \sim 1. \tag{4.2}$$

Further, when χ_n/kT_e is small

$$-Ei(-\chi_n/kT_e) \sim -y - \log_e \chi_n/kT_e, \tag{4.3}$$

where y is Euler's constant, 0·5772. This form shows that when χ_n/kT_e is small $-Ei(-\chi_n/kT_e)$ is fairly insensitive to changes in T_e. Consider, for example, the level $n=25$. The value of $-Ei(-\chi_n/kT_e)$ changes from (2.4) to (4.7) as T_e changes from 5000° to 50,000°. Neglecting therefore the comparatively small change in $-Ei(-\chi_n/kT_e)$ with T_e, we find that the number of captures in state n, when n is large, is proportional to

$$\frac{1}{n^3} \frac{N_i N_e}{T_e^{3/2}}. \tag{4.4}$$

Further, according to (3.2), the emission per unit wave-number in the Balmer continuum at the limit is also proportional to $\frac{N_i N_e}{T_e^{3/2}}$. But the emission in the upper members of the Balmer series depends directly on the captures in the various upper states. Hence in a capture spectrum the intensities of the high members of the Balmer series, relative to the intensity in the Balmer continuum per unit wave-number at the series limit, should show very little change with temperature.

This conclusion appears anomalous at first sight, because we know that for increasing electron temperature, T_e, the numbers of electrons captured in the lower states increase relative to those captured in the higher states. Therefore, relative to the emission in the Balmer continuum as a whole, the discrete Balmer spectrum near the limit becomes weaker with increasing T_e. Since, however, the intensity in a frequency range $d\nu$ of the continuum is proportional to

$$e^{-h\nu/kT_e} d\nu, \tag{4.5}$$

contributions to this emission tend to come more from the higher frequencies for high values of T_e. This is due to the capture of a greater proportion of high energy electrons. The number of low energy electrons captured on the second level—those which give rise to the Balmer continuum near the limit—relative to the total numbers captured in the upper states ($n > 20$) does not change appreciably with T_e.

In view of the continuity in the absorption coefficients for the hydrogen states of low negative and low positive energies,* self-reversal will affect

* Sugiura, *Sci. Papers Inst. of Phys. and Chem. Res.*, **11**, 1, 1929.

upper members of the Balmer series and the Balmer continuous spectrum near the limit in the same way. We conclude that E_n/Ba_c ($n > 20$) should be approximately the same, whatever the electron temperature or the degree of self-reversal.

In Table V are given Page's measurements of the intensity in the spectra of certain planetary nebulæ at two wave-lengths, λ 3650 and λ 3680.* Because of the wide slit used in obtaining the spectra, about 40 A. at λ 3650, the members of the series near the limit overlap to form a continuous spectrum. The intensity in this continuous spectrum at λ 3650 and λ 3680, however, is some indication of the intensities of the Balmer continuous spectrum at the limit and of the Balmer lines in the neighbourhood of λ 3680. Columns 3 and 4 give the intensities as measured, and column 5 gives their ratio, which does not vary over a large range.

TABLE V

Observed Spectra of Planetary Nebulæ near the Balmer Limit (Page)

Object	V_c	I_λ (uncorrected)			I_λ (corrected)		
		λ 3680	λ 3650	I_{3650}/I_{3680}	λ 3680	λ 3650	I_{3650}/I_{3680}
N.G.C. 7662 (west side)	1·00	1·04	1·46	1·40	0·04	0·46	11·50
N.G.C. 7662 (east side)	1·12	1·26	2·26	1·79	0·14	1·14	8·15
N.G.C. 6572	1·12	1·81	2·62	1·45	0·69	1·50	2·18
N.G.C. 7027	1·12	1·12	1·78	1·59	0·00	0·66	∞
I.C. 4593	1·00	1·08	1·24	1·15	0·08	0·24	3·00

However, Page finds that the line emission of the nebulæ is superposed on a general emission, whose intensity V_c, given in column 2, is constant over the range λ 3927 to λ 4800. Applying this correction for background also to his measured intensities in the ultra-violet, he finds the intensities given in columns 6 and 7 of Table V. The ratio of the corrected values of I_{3650} and I_{3680} in column 8 completely disagrees with the above theoretical prediction. Since, however, the intensities of the background at λ 3680 and λ 3650 are found by extrapolation, it is possible that the value of V_c represents an over-correction.

5. *Conclusion.*—Menzel and Pekeris' recent values for the Einstein transition coefficients for hydrogen are used to derive the relative intensities of the Balmer lines up to H_{10} in a capture spectrum. The result of a previous

* Page, *loc. cit.*, Tables IV, V and VI.

investigation is confirmed, namely, that the theory fails to explain the observed difference in the Balmer decrement ($H\alpha$, $H\beta$ and $H\gamma$) for different nebulæ as due to a difference in electron temperature. There is also close agreement between the present values of the relative intensities of $H\alpha$, $H\beta$ and $H\gamma$ and those derived before. We conclude that the theoretical intensities of the early members of the series are not appreciably affected by the upper states.

The intensities of the lines, E_n, relative to the intensity per unit wave-number in the Balmer continuous spectrum at the limit are given. These quantities appear to be more sensitive to the value of the electron temperature than the Balmer decrement. They are compared with observations of the chromosphere and show the closest agreement for an electron temperature of 10,000°. Comparison with Page's observations of the continuous spectra of planetary nebulæ shows that the computed intensities of the Balmer lines, relative to the intensity in the Balmer continuum, are about a hundred times weaker than the observed ones. This shows that the simple conditions, assumed in the calculation, do not obtain in the nebulæ. If self-reversal of the Balmer spectrum in nebulæ is the cause of this discrepancy, Page's observations indicate that there is considerable self-reversal for the first few members of the series.

It is found that the intensities of the Balmer lines near the limit relative to the intensity of the continuous spectrum at the limit should show little variation with the electron temperature or with the amount of self-reversal. This does not agree with Page's observations for planetary nebulæ, although it is possible that he has over-corrected his Balmer intensities in the ultra-violet for the general background emission. If this is the case, part of the difference would be removed.

In conclusion it is a pleasure to record my thanks to Professor H. H. Plaskett for helpful criticisms and suggestions. Also to Mr. T. L. Page for suggestions and for kindly communicating his observational results in advance of publication.

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NOTE ON THE DERIVATION OF THE EQUATION OF STATE FOR A DEGENERATE RELATIVISTIC GAS.

R. Peierls.

(Communicated by S. Chandrasekhar)

The problem of a degenerate relativistic gas has recently become of importance for work on stellar structure.* Some controversy has arisen as to whether there is an equation of state in the usual sense of the word, *i.e.* whether the pressure-density relation of such a gas enclosed in a certain volume would be independent of the shape of this volume.† This might seem sufficiently obvious to make a proof unnecessary, but in view of the controversy it is perhaps worth while to give a proof.

We shall assume that the present form of quantum mechanics applies to the problem, and only prove that from this theory one obtains the usual equation of state.

Further, we shall neglect the interaction between the particles and thus we may restrict ourselves to treating the one-body problem. We shall determine the number of possible states of motion of one electron in a region of volume V with energies between E and $E+dE$, say

$$dZ = N(E)dE\,; \tag{1}$$

from this the equation of state follows by a straightforward application of Fermi's statistics.

It is usually stated that the possible states of motion are evenly distributed in momentum space and that there are

$$2h^{-3}V dp_x dp_y dp_z, \tag{2}$$

states with momentum components between p and $p+dp$. Here h is Planck's constant, V the volume, and the factor 2 is due to the two possible spin directions. Combining (2) with

$$E^2 = p^2c^2 + m^2c^4 \tag{3}$$

one obtains for the expression (1):

$$N(E) = 8\pi(hc)^{-3}VE(E_0^2 - m^2c^4)^{\frac{1}{2}}. \tag{4}$$

The justification for (2) is usually given by showing that it is correct if, instead of enclosing the electron in a region of volume V, one imposes periodic boundary conditions for its wave function, such that the wave function is known in all space if its values are given in a region of volume V.‡ It might, however, be argued that such boundary conditions can never in

* Chandrasekhar, *M.N.*, **95**, 207, 226, 676, 1935.

† Eddington, *Proc. Roy. Soc.*, A, **152**, 258, 1935.

‡ *Cf.* Møller and Chandrasekhar, *M.N.*, **95**, 673, 1935.

practice be realized, and we shall therefore discuss the case of an electron moving in a potential field which confines it to the inside of a region R of volume V with a shortest linear extension of L. We assume a potential function $U(xyz)$ with the following properties : (1) $U=0$ in the interior of R. (2) U tends to a finite value, U_0, at large distance. U_0 must be greater than the kinetic energy of the electrons we are concerned with. (3) U rises from 0 to U_0 in a boundary layer of thickness d on the surface of R. We assume $d << L$, while d is sufficiently large to exclude pair creation (see below).

The wave function of the electron in the nth state of motion has four components, $\psi_{n\rho}(\rho=1, 2, 3, 4)$ which satisfy Dirac's equation :

$$E_n\psi_{n\rho} - c\sum_{k=1}^{3}\sum_{\rho'=1}^{4}\alpha^k_{\rho\rho'}p_k\psi_{n\rho'} - mc^2\sum_{\rho'}\beta_{\rho\rho'}\psi_{n\rho'} = 0, \tag{5}$$

where p_k denotes, as usual, the operator $-i\hbar\partial/\partial x_k(\hbar = h/2\pi)$. α^k and β are Dirac's spin matrices.

The function ψ_n is normalized :

$$\sum_{\rho}\int d\tau \mid \psi_{n\rho}(xyz) \mid^2 = 1. \tag{6}$$

In order to determine the number of states per unit energy in the neighbourhood of E_0 we choose a quantity ΔE with the following properties :—

(a) $\Delta E << E_0$.

(b) $hv/\Delta E << L$, where v is a function of E_0, defined by $E_0 = mc^2(1 - v^2/c^2)^{-\frac{1}{2}}$.

Condition (b) is equivalent to the requirement that the simultaneous knowledge of the position within L and of the kinetic energy within ΔE are not in contradiction with the uncertainty principle. Our results are valid only if (a) and (b) are compatible, *i.e.* if

$$hv/LE_0 << 1.$$

This condition means that there must be a great number of energy levels below E_0.

Next we choose a weight function $D(E)$ with the properties

$$\text{(A)}, \qquad D(E) = 0 \text{ if } E > \Delta E.$$

$$\text{(B)}, \qquad \int_{-\infty}^{\infty} D(E)dE = \Delta E. \tag{7}$$

This function can then be expressed as a Fourier integral :

$$D(E) = \int f(s)e^{-iEs/\hbar}ds, \tag{8}$$

and we determine D in such a way that in this integral

(C), only values of s of the order $\hbar/\Delta E$ give an appreciable contribution. This is compatible with (A).

Using this function, we can write

$$N(E_0)\Delta E = \sum_n D(E_n - E_0),$$

where the sum goes over all possible states. Or, from (6):

$$N(E_0)\Delta E = \sum_n D(E_n - E_0) \sum_\rho \int d\tau \psi^*_{n\rho} \psi_{n\rho}$$
$$= \sum_\rho \int d\tau (xyz\rho \mid D(E - E_0) \mid xyz\rho), \qquad (9)$$

where in the last line we have introduced the symbol

$$(xyz\rho \mid D(E - E_0) \mid x'y'z'\rho') = \sum_n D(E_n - E_0) \psi_{n\rho}(xyz) \psi^*_{n\rho'}(x'y'z'). \qquad (10)$$

We shall now show that the integrand of (9) is a constant, *i.e.* independent of xyz for all points in the interior of R, except possibly for a boundary layer of thickness $hv/\Delta E$. Since, however, ΔE was chosen to satisfy the condition (2), the contribution of this layer to the integral may be neglected.

In the following we calculate the value of the integrand for a point whose distance from the surface is greater than $hv/\Delta E$, and show that it is identical with what one would obtain for free electrons, viz.

$$(xyz\rho \mid D(W - E_0) \mid x'y'z'\rho') = (2\pi)^{-3} \sum_{\zeta=1,2} \int dk_x dk_y dk_z e^{ik(x-x')}$$
$$\{a_{\rho\zeta} a^*_{\rho'\zeta} D(W - E_0) + a_{\rho,-\zeta} a^*_{\rho',-\zeta} D(-W - E_0)\}. \qquad (11)$$

Here $W(k)$ is $c(\hbar^2 k^2 + m^2 c^2)^{\frac{1}{2}}$ and $a_{\rho\zeta}$ are a set of four numbers such that

$$\psi_\rho = a_{\rho\zeta} e^{ikx}$$

satisfies the wave equation (5) with $U = 0$ and the energy $+W$, while $a_{\rho,-\zeta}$ are the corresponding coefficients for the energy $-W$. There are two independent solutions for given energy; we distinguish them by $\zeta = 1$ and $\zeta = 2$. $a_{\rho\zeta}$ is normalised:

$$\sum_\rho |a_{\rho\zeta}|^2 = 1.$$

We now remember the form (8) for D. Since both (10) and (11) are linear, their identity will be proved if we can show that the same holds for the integrand of (8), in other words, that for values of s of the order $\hbar/\Delta E$, the expressions

$$(xyz\rho \mid e^{-iEs/\hbar} \mid x'y'z'\rho') = \sum_n \psi_{n\rho}(xyz) \psi^*_{n\rho'}(x'y'z') e^{-iE_n s/\hbar} \qquad (12)$$

and

$$(xyz\rho \mid e^{-iWs/\hbar} \mid x'y'z'\rho') = \sum_{\zeta=1,2} (2\pi)^{-3} \int dk_x dk_y dk_z \{a_{\rho\zeta} a^*_{\rho'\zeta} e^{-iWs/\hbar}$$
$$+ a_{\rho,-\zeta} a^*_{\rho',-\zeta} e^{+iWs/\hbar}\} e^{ik(x-x')} \qquad (13)$$

are identical. This follows at once if one takes into account a characteristic property of the expressions (12) and (13).

Consider an arbitrary solution of the time dependent wave equation which, at $t = 0$, has the form $\psi_\rho(xyz\psi\, 0)$. Then the value of ψ at any other time t is given by

$$\psi_\rho(xyzt) = \sum_{\rho'} \int d\tau' (xyz\rho \mid e^{-iEt/\hbar} \mid x'y'z'\rho') \psi_{\rho'}(x'y'z'0), \qquad (14)$$

using the definition (12). This follows because (14) is obviously a solution of the wave equation and because it becomes equal to $\psi_\rho(xyz0)$ at $t=0$. The latter statement is obvious from the orthogonality and normalization of the functions $\psi_{n\rho}$ if ψ_ρ can be expanded in terms of the system of functions $\psi_{n\rho}$. It is thus generally true, as it is well known that every function of sufficient regularity can be expanded in this way.

If, on the other hand, we consider the motion of a *free* electron with no potential field, assuming the same initial wave function conditions, an analogous argument shows that the wave function at time t will be

$$\phi_\rho(xyzt) = \sum_{\rho'} \int d\tau' ((xyz\rho) e^{-iWt/\hbar} \mid x'y'z'\rho') \phi_{\rho'}(x'y'z'0), \tag{15}$$

using the definition (13).

In particular we may choose a wave function which at $t=0$ is different from 0 only at points in the interior of R that have a distance greater than $\hbar v/\Delta E$ from the surface, but is otherwise arbitrary. Then we are sure that initially the particle moves in a region where $U=0$ and where, consequently, the presence or absence of the potential field makes no difference to the motion of the particle. This holds as long as we are sure that the particle has not reached the surface yet, *i.e.* until the time t becomes larger than $\hbar/\Delta E$. Thus for times t of the order $\hbar/\Delta E$, (14) and (15) must be equal, and since $\psi_\rho(xyz\ 0)$ is arbitrary except for vanishing near the boundary and outside, it follows that also (12) and (13) must be equal if both points xyz and $x'y'z'$ are not near the surface or outside.

Hence also (10) and (11) must be equal for such points, and consequently we may insert (11) instead of (10) into the integral (9) if we neglect surface terms. The expression

$$\sum_\rho (xyz\rho \mid D(W-E_0) \mid xyz\rho)$$

can be evaluated in an elementary way and yields :

$$\frac{8\pi}{(2\pi)^3} \int \frac{dk}{dW} k^2 D(W-E_0) dW.$$

Since $\Delta E << E_0$ all factors in the integrand besides D are constant over the range ΔE, and we thus obtain because of (7) :

$$\frac{8\pi}{(2\pi)^3} \left[\frac{dk}{dW} k^2\right]_{W(k)=E_0} . \Delta E = \frac{8\pi}{(\hbar c)^3} E_0 \sqrt{E_0^2 - m^2c^4} . \Delta E.$$

The integration in (9) contributes a factor V as the integrand is constant and we therefore obtain indeed the result (4).

The proof is therefore complete but for the discussion of a possible difficulty. We have assumed that there exists a potential field of the kind described which will actually confine the electron to the inside of the region R. This assumption wants some explanation, for if the energy E_0 is greater

than $2mc^2$, U_0 must also exceed $2mc^2$ in order to stop the electron. This means that for electrons of *small* velocity for which $E \sim mc^2$, we have

$$E - U < -mc^2 \quad \text{outside } R.$$

Now the Dirac equation has solutions behaving like plane waves in a region where this condition is fulfilled, and thus slow electrons could escape from the potential field, their kinetic energy becoming negative. In other words, we cannot use the same potential field in order to confine electrons of different energies to within R.

However, as Sauter and Pauli * have shown, the probability of this process is only appreciable if the thickness b of the layer in which the potential rises by $2mc^2$ is comparable to $b_0 = \hbar/mc \sim 3 \cdot 10^{-11}$ cm. Otherwise the transmission coefficient of the boundary layer becomes of the order of

$$e^{-\pi b/b_0}.$$

Thus even if the thickness of the boundary layer is of the order of 1 cm., which is well compatible with our other conditions for d, provided $> > 1$ cm., the transmission coefficient becomes less than

$$e^{-10^{11}}. \tag{16}$$

Hence we can work with wave functions that are not exact solutions of (5), *i.e.* do not belong to exactly defined energy values, but instead have, at $t = 0$, appreciable values only inside R. For times during which the probability of escape remains small, these functions have then all the properties of solutions of (5), *i.e.* they are orthogonal and complete except for terms of the order (16). The times during which this is admissible are, according to (16), large even on the astronomic scale. For the same reason the uncertainty in the energy which arises from this cause is negligible.

Actually the states of negative kinetic energy should be eliminated with the aid of Dirac's theory of the positron. This makes the escape of electrons impossible, but instead introduces the possibility of creation of a pair, the positive electron going off to the outside, the negative to the inside, of R. However, the probability of this process is also governed by the factor (16) and may therefore be neglected.

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* Sauter, *Z. Physik*, 69, 742, 1931. Pauli, *Helv. Phys. Acta*, 5, 171, 1932.

IONIZATION AND EXCITATION IN THE UPPER LAYERS OF AN ATMOSPHERE.

A. Pannekoek.

The Saha ionization formula has been deduced for the case of a gas in thermodynamical equilibrium. In a stellar atmosphere this condition is not fulfilled; so it may be of interest to see what corrections must be applied to results deduced from the Saha formula. For the case of ionization by radiation (we may assume that ionization by collisions does not play an important part in the relevant atmospheric layers) Milne has deduced a formula containing probability factors for ionization through radiation of various frequencies. The factor by which the Saha result for the temperature $K = Px/(1-x)$ has to be multiplied is *

$$\frac{\int \mathcal{J}\psi d\nu}{\int \psi e^{-h\nu/kT}\left(\frac{2h\nu^3}{c^2}+\mathcal{J}\right)d\nu},$$

where $4\pi\mathcal{J}$ is the total radiation of frequency ν falling from all sides on the atoms, ψ is the probability factor of ionization by ν radiation (*i.e.* the number of ionizations produced by $\mathcal{J}(\nu)d\nu$ is $\mathcal{J}(\nu)\psi(\nu)d\nu$ per atom), T is the gas temperature dominating the recombinations; the integration extends over all ν from 0 to ∞. The first denominator term denotes the spontaneous recombinations, the second term denotes the stimulated ones, introduced by Milne. The probability factor $\psi(\nu)$ is connected with the absorption coefficient $k(\nu)$ by the relation $h\nu\psi = k$, because at every ionization a quantum $h\nu$ is absorbed.

In *B.A.N.*, 110,† this formula was applied to the case of black-body radiation at the effective temperature T_1 of the star, coming from below and ionizing the surface layer of the atmosphere, where the gas temperature is T_0. The result was, apart from a factor $\frac{1}{2}$ for the dilution of the radiation, the replacement of the temperature T_0 by T_1 in the exponential function in Saha's formula. In the same way in the Boltzmann factor for the number of excited atoms the effective temperature T_1 had to be used instead of the gas temperature T_0. This is in accordance with the result afterwards found by Russell‡ that in order to represent the observations of line intensities for the Sun the logarithms of the computed fractions of atoms in different states of excitation must be multiplied by 0·85, whereas $T_0/T_1 = 0{\cdot}83$.

These results could only be a second approximation, because now we know that the radiation passing through the upper layers of an atmosphere and emitted at the surface is not black-body radiation. By knowledge of the absorption coefficients we are able to compute the intensity distribution in the emitted spectrum. So with these improved values of $\mathcal{J}(\nu)$, which

* Milne, *Phil. Mag.*, **47**, 209; Woltjer, *Physica*, V, 406; Pannekoek, *B.A.N.*, 110.
† *B.A.N.*, **3**, 207, 1926. ‡ *Astroph. J.*, **70**, 53, 1929.

thus are based entirely on theoretical computation, we may compute the rate of ionization to a further approximation. We will first consider the high temperature stars, because for these the absorption coefficient is determined entirely by the hydrogen atoms and it is permissible to treat the case as being one of a pure hydrogen atmosphere.

Hydrogen Atmosphere.—The hydrogen absorption coefficient decreases with increasing wave-length between two consecutive band edges as ν^{-3}, but then at the next lower band edge jumps up to a higher value. So do the probability coefficients ψ, which are highest just below each band edge, decrease as ν^{-4} and jump at the next band edge to a high value again. The radiation, however, shows a dark absorption edge, hence has a small intensity precisely at the wave-lengths where k and ψ are high. So there is a certain compensation of the factors occurring in the integrals.

We consider only the surface layers where for $\mathcal{J}$ we may take the surface value of the radiation

$$\mathcal{J}(\nu)/E(\nu, 0) = \frac{2}{2+\sqrt{3}}\left(1+\frac{\sqrt{3}}{2}c\frac{\bar{k}}{k}\right),$$

where $4\pi E(\nu, 0)$ is the black-body radiation for the surface temperature and $c = \frac{1}{4}h\nu/kT(1-e^{-h\nu/kT})^{-1}$. The absorption coefficients $k(\nu)$ and $\bar{k}$ occurring in this expression comprise the total absorption due to free-free transitions as well as to ionizations. To distinguish them we will denote the absorption coefficients due solely to ionization, to which are related the probability functions ψ, by k'. The factor $2/(2+\sqrt{3})$ will be replaced by $1/2$. These absorption coefficients k and k' will be represented by

$$k = k_n(\nu_0/\nu)^3, \qquad k' = k_n'(\nu_0/\nu)^3,$$

where for ν_0 is taken the value for some standard wave-length $\lambda = 4000$ A., and k_n and k_n' are constant between two band edges, jumping from the highest value k_1 between ν_1 and ∞, k_2 between ν_2 and ν_1, down at every next band edge. For $E(\nu, 0)$ we take the Planck function for the surface temperature (which we write T instead of T_0) and put $h\nu/kT = y$.* We omit some coefficients which are the same in numerator and denominator, viz. $2h/c^2$ from the Planck function, and h from the value $\psi = k'/h\nu = (\nu_0/\nu)^3 k'_n/h\nu$. Then the correction factor required is

$$\frac{\int \frac{1}{2}\nu^3(e^y-1)^{-1}\left(1+\frac{\sqrt{3}}{2}c\frac{\bar{k}}{k}\right)\frac{k'}{\nu}d\nu}{\int \nu^3 e^{-y}\frac{k'}{\nu}d\nu + \int \frac{1}{2}\nu^3 e^{-y}(e^y-1)^{-1}\left(1+\frac{\sqrt{3}}{2}c\frac{\bar{k}}{k}\right)\frac{k'}{\nu}d\nu}.$$

Each integral extends from 0 to ∞; because of the jumps in k and k' we divide it into an infinite number of integrals between the successive band edges. For the term with c, however, this is not necessary; because k and k' are varying in the same way, we may keep it as one integral between

* The symbol k for the absorption coefficient has to be used also for the Boltzmann constant. Since in the latter case it always occurs in the combination kT no difficulties need to arise.

o and ∞, with (k'/k) as a factor in the integrand. For the numerator we may write now

$$N = \int \tfrac{1}{2} k_n' \nu_0^3 (e^y - 1)^{-1} \frac{d\nu}{\nu} + \frac{\sqrt{3}}{4} \bar{k} \int c\left(\frac{k'}{k}\right) \nu^2 (e^y - 1)^{-1} d\nu = N_1 + N_2,$$

$$N_1 = \tfrac{1}{2} k_1' \nu_0^3 \int_{y_1}^{\infty} \frac{dy}{y(e^y - 1)} + \tfrac{1}{2} k_2' \nu_0^3 \int_{y_2}^{y_1} \frac{dy}{y(e^y - 1)} + \ldots,$$

$$N_2 = \frac{\sqrt{3}}{4} \bar{k} \left(\frac{kT}{h}\right)^3 \int_0^{\infty} c\left(\frac{k'}{k}\right) \frac{y^2 dy}{e^y - 1}.$$

In each of the terms of N_1 the constant k_n' is given by an infinite sum

$$k_n \nu_0^3 = \sum_n^{\infty} C_1 n^{-3} e^{-h\nu_1/n^2 kT},$$

where C_1 is composed of constants.* For each of the first terms this expression can be computed separately as the sum total of a convergent series. For the higher terms constituting an infinite tail, where y in each term changes little from y_{n-1} to y_n, the sum may be replaced by an integral (extending from $n - \frac{1}{2}$ to ∞, for which we may use as well the limits n to ∞)

$$k_n' \nu_0^3 = \int_n^{\infty} C_1 n^{-3} e^{-h\nu_1/n^2 kT} dn = C_1 \frac{kT}{h\nu_1} (e^{y_n} - 1),$$

so that the successive higher terms of N_1 of the form

$$\tfrac{1}{2} k_n' \nu_0^3 \int_{y_n}^{y_{n-1}} \frac{dy}{y(e^y - 1)} = \tfrac{1}{2} \frac{C_1}{y_1} \int_{y_n}^{y_{n-1}} \frac{dy}{y}$$

can be collected into one integral from $y = o$ to $y = y_n$. So we have

$$N_1 = \tfrac{1}{2} \sum_1^n k_n' \nu_0^3 \int_{y_n}^{y_{n-1}} \frac{dy}{y(e^y - 1)} + \tfrac{1}{2} \frac{C_1}{y_1} \int_0^{y_n} \frac{dy}{y}.$$

Thus N_1 is logarithmically infinite. This was to be expected, because it is well known that the Boltzmann distribution of hydrogen levels forms a non-convergent infinite series. In the partition function

$$\Sigma g_n \exp\left(-h\nu_1(1 - 1/n^2)/kT\right)$$

the exponential function with increasing n converges to a small constant fraction of the first term. The difficulty has been removed by Urey and by Fermi,† who took into account the circumstance that for atoms occupying a limited volume n cannot increase indefinitely, and the higher levels are gradually cut down by a volume function. Hence we are entitled to leave out the infinite number of highest levels, and we will do it by cutting off the tail at $n = \omega$, some big number, so that the limit o is replaced by y_ω. Then the last part of N_1 will be

$$\tfrac{1}{2} \frac{C_1}{y_1} (ln y_n - ln y_\omega) = \tfrac{1}{2} \frac{C_1}{y_1} ln \frac{\omega^2}{n^2} = \frac{C_1}{y_1} ln \frac{\omega}{n}.$$

* *Publ. Amsterdam*, IV, pp. 12, 13.

† *Astroph. J.*, 59, 1, 1924; *Z. f. Physik*, 26, 54, 1924; cf. *Handb. d. Astrophysik*, III, 1, 270.

Here y_1 corresponds to the lowest level of hydrogen, n is the number of the level where the separate terms cease and ω is some big number.

We have now to add N_2, the part of the numerator integral which is due to the increase of temperature in deeper layers. For the factor c we put its value $\frac{1}{4}y(1-e^{-y})^{-1}$. The ratio k'/k is nearly 1 for large ν, where the absorption through free-free transitions is small compared with ionization from the lowest levels. For small ν the free-free transitions play a dominant part. An approximate value for k'/k may be found if we assume for it the value it takes in the case of a smoothed absorption coefficient of a mixture with numerous band edges. In this case $k'/k = 1 - e^{-y}$; if we use this expression over the whole range of ν we get

$$N_2 = \frac{\sqrt{3}}{16}\bar{k}\left(\frac{kT}{h}\right)^3\int_0^\infty \frac{y^3dy}{e^y-1} = \frac{\sqrt{3}}{16}\cdot\bar{k}\left(\frac{kT}{h}\right)^3\frac{\pi^4}{15}.$$

We now turn to the denominator integrals, the first of which represents the effect of the spontaneous captures. Just as in N_1 it is divided into a number of integrals between the successive band edges, whereas the sum total of all the tail terms from the nth term onward is collected into a single integral. The integrands differ from those in N_1 in that $(e^y-1)^{-1}$ is replaced by e^{-y}. So we have

$$D_1 = \sum_1^n k_n'\nu_0^3\int_{y_n}^{y_{n-1}}\frac{e^{-y}dy}{y} + \frac{C_1}{y_1}\int_0^{y_n}\frac{dy}{y}(1-e^{-y}).$$

For the second of these integrals there are no difficulties at the zero limit, because both the logarithm and the exponential integral tend to infinity and their difference tends to the Eulerian constant $\gamma = 0{\cdot}5772$. So for it we write

$$\frac{C_1}{y_1}\{lny_n + Ei(-y_n) + \gamma\}.$$

The other denominator integral D_2, produced by the stimulated recombinations, has the same form as the numerator N, with only the factor e^{-y} added. The effect of this factor is that for the first regions of ν, corresponding to the lowest levels, the integrals of D_2 are much smaller than those of N. For the higher levels, however, for $y \to 0$ the infinite tail of later terms approaches to the same value as in N. It likewise gives rise to a logarithmic infinity, which must be avoided by the introduction of the limit ω. This means that the transitions with low energy difference between very high levels and the ionized state depend, in either direction, chiefly on the low frequency radiation, whereas the one-sided spontaneous transitions are insignificant. We have

$$D_2 = \tfrac{1}{2}\sum_1^n k_n'\nu_0^3\int_{y_n}^{y_{n-1}}\frac{e^{-y}dy}{y(e^y-1)} + \tfrac{1}{2}\frac{C_1}{y_1}\int_0^{y_n}\frac{e^{-y}dy}{y} + \frac{\sqrt{3}}{16}\bar{k}\left(\frac{kT}{h}\right)^3\int_0^\infty\frac{y^3e^{-y}dy}{e^y-1}.$$

Each of the first integrals can be reduced

$$\int\frac{e^{-y}dy}{y(e^y-1)} = \int\frac{dy}{y(e^y-1)} - \int\frac{e^{-y}dy}{y},$$

so that $D_2 = N - \frac{1}{2}D_1$, or $D_1 + D_2 = N + \frac{1}{2}D_1$. This holds in the same way for the sum total of the tail terms. For the third part of D_2, produced by the temperature increase in the deeper layers, we have

$$\int_0^\infty \frac{y^3 e^{-y} dy}{e^y - 1} = \frac{\pi^4}{15} - 6 = 0{\cdot}4940.$$

For the numerical computation of the separate terms in N_1 integrals are needed of the form $\int_y^\infty \frac{dy}{y(e^y - 1)}$. They have been computed by Dr. J. R. Airey who kindly provided me with a manuscript copy of them. We will indicate them by $A(y)$. In D_1 the terms can be found as differences of exponential integrals $-Ei(-y)$.

TABLE I

Correction Factor $\frac{N}{D}$ *to the Saha Ionization Formula for Hydrogen Stars*

$$\frac{x}{1-x} P = K \frac{N}{D}$$

	25200°		16800°		12600°		10080°	
	Num.	Den.	Num.	Den.	Num.	Den.	Num.	Den.
1st term	107	215	205	410	256	512	460	920
2nd ,,	58	108	100	196	151	298	210	418
3rd ,,	44	72	80	140	120	220	163	308
4th ,	34	47	62	94	94	154	130	223
5th ,,	27	34	49	67	76	110	105	160
6th ,,	22	26	40	51	62	84	87	122
Term with $\bar{k}$	15	1	32	2	85	6	169	13
Tail for N and D_2 $\omega = 30$	197		363		557		778	
$\omega = 60$	282		518		796		1112	
$\omega = 120$	367		673		1035		1446	
Tail of D_1	20		55		110		188	
$N : D$	504 : 720		931 : 1378		1401 : 2051		2102 : 3130	
,,	589 : 805		1086 : 1533		1640 : 2290		2436 : 3464	
,,	674 : 890		1241 : 1688		1879 : 2529		2770 : 3798	
N/D	0·70		0·68		0·68		0·67	
,,	0·73		0·71		0·72		0·70	
,,	0·76		0·74		0·74		0·73	

The computations have been made for temperatures 25200°, 16800°, 12600° and 10080°; the first six integrals were computed separately, and the sum total was taken of the tail from the 6th level downward. For the large number ω where the tail was cut off, three values 30, 60, 120 were taken, hence $\omega/n = 5$, 10, 20. The limiting values $y_1 \ldots y_6$ of the integration domains are found by $(2{\cdot}3026 \times 13{\cdot}54)\ (5040/T)$ divided by 1, 4, . . . 36. The results are given in Table I. The separate parts of numerator and denominator are given there in unit 0·00001. The correction factor for all these cases comes out about 0·70.

The difference between this result and the former result of *B.A.N.*, 110,

can easily be traced by means of the separate terms. In the former treatment only ionizations from the lowest level were considered. Here we see that for hydrogen the contributions of the higher levels decrease, but slowly, and if the number of levels actually occupied rises above 20 or 30 (which is certainly the case for giant and supergiant stars) the contribution of all these higher levels begins to exceed that of the lowest level. For the lowest level $N_1/D_1 = \frac{1}{2}$; for the highest levels, where both ionization and recombination are chiefly determined by the radiation $N_1/D_2 = 1$. So with the increasing importance of the higher levels the ratio goes up from $\frac{1}{2}$ to 1.

The other difference is due to the increase of temperature in the deeper layers. The term produced by this increase (the term with $\bar{k}$) is unimportant in the present computation. The bright parts of the spectrum (at the long wave side of the band edges), where light from deep layers is emitted, have a small absorption coefficient and a small ψ, hence they contribute little to the ionization, whereas the parts that could contribute most are dark and their radiation comes only from the superficial layers. In the former computation the emitted light was supposed to be unweakened black-body radiation, all coming from rather deep layers. If we introduce the former suppositions into the present formulæ, hence taking $\mathfrak{J} = E_0(1+c)$ and using only the lowest level, we have

$$\frac{N}{D_1+D_2} = \frac{\frac{1}{2}\int \frac{dy}{y(e^y-1)} + \frac{1}{8}\int \frac{e^y dy}{(e^y-1)^2}}{\int \frac{e^{-y}dy}{y} + \frac{1}{2}\int \frac{e^{-y}dy}{y(e^y-1)} + \frac{1}{8}\int \frac{dy}{(e^y-1)^2}},$$

all integrals being taken from y_1 to ∞, where y_1 is always a large number $(31{\cdot}18 \times 5040/T)$. The first integral in the numerator is Airey's integral, the first in the denominator the exponential integral, the second in the denominator is the difference of these two, which is imperceptible for high values of y_1. From the two integrals representing the temperature increase with depth the second numerator term $= \frac{1}{8}(e^{y_1}-1)^{-1}$, the third denominator integral $= \frac{1}{8}((e^{y_1}-1)^{-1} - ln(1-e^{-y_1})) = \frac{1}{8}(\frac{1}{2}e^{-2y_1} + \frac{3}{4}e^{-3y_1} + \ldots)$. Numerical computation gives for 25200° ($y_1 = 6{\cdot}235$), and for 10080° ($y_1 = 15{\cdot}59$):

$$\frac{N}{D} = \frac{0{\cdot}14 \times 10^{-3} + 0{\cdot}25 \times 10^{-3}}{0{\cdot}28 \times 10^{-3} + 0{\cdot}3 \times 10^{-6} + 0{\cdot}24 \times 10^{-6}} = 1{\cdot}4,$$

$$\frac{N}{D} = \frac{0{\cdot}52 \times 10^{-8} + 0{\cdot}21 \times 10^{-7}}{1{\cdot}04 \times 10^{-8} + (10^{-14})} = 2{\cdot}5.$$

Here the second numerator term is even larger than the first and shows the influence of the deep hotter layers, which in the old computation was expressed by the replacement of the surface temperature T_0 by the effective temperature T_1.

There can be no doubt that the new computation, showing a very small influence of the temperature increase only, is more correct. We have to

consider, however, that our computation relates to the uppermost layers; in the deeper layers the radiation comes from all sides and the temperature is higher; so the ionization computed with the old formula may be nearly right for the mean of the relevant atmospheric layers.

The Solar Atmosphere.—The solar atmosphere is taken here as the type of a low-temperature atmosphere where the absorption is determined by a mixture of a large number of elements (metals). We assume that the number of levels and band edges is so large that for the dependence of the absorption coefficient on wave-length we may take a smoothed function. Taking the Gaunt factors for the free-free transition as well as for the ionizations 1, we have

$$k' = C_0\left(\frac{\nu_0}{\nu}\right)^3 (e^{h\nu/kT} - 1); \qquad k = C_0\left(\frac{\nu_0}{\nu}\right)^3 e^{h\nu/kT}.$$

We will consider the ionization of an element that has such a large number of levels that for the ionization probability ψ we may take the same smoothed form of k' as for the total mixture. Then we have

$$\psi = C_0 \frac{{\nu_0}^3}{h\nu^4}(e^{h\nu/kT} - 1); \qquad \mathfrak{J} = \tfrac{1}{2}E_0\left(1 + \frac{\sqrt{3}}{2}c\frac{\bar{k}}{\bar{k}}\right).$$

Omitting the same common factors as in the previous case we write for the correction factor

$$\frac{C_0{\nu_0}^3\int \tfrac{1}{2}\nu^3(e^y-1)^{-1}\frac{d\nu}{\nu^4}(e^y-1) + \frac{\sqrt{3}}{16}\bar{k}\int \nu^3(e^y-1)^{-2}\frac{d\nu}{\nu}y\left(\frac{k'}{k}\right)}{C_0{\nu_0}^3\int \nu^3 e^{-y}\frac{d\nu}{\nu^4}(e^y-1) + C_0{\nu_0}^3\int \tfrac{1}{2}\nu^3 e^{-y}(e^y-1)^{-1}\frac{d\nu}{\nu^4}(e^y-1) + \frac{\sqrt{3}}{16}\bar{k}\int \nu^3 e^{-y}(e^y-1)^{-2}\frac{d\nu}{\nu}y\left(\frac{k'}{k}\right)}.$$

Considering that $\bar{k} = C_0\left(\frac{h\nu_0}{kT}\right)^3 \frac{1}{0{\cdot}873}$ * and putting $\frac{k'}{k} = 1 - e^{-y}$, we find

$$\frac{\int \tfrac{1}{2}\frac{dy}{y} + \frac{\sqrt{3}}{16}\frac{1}{0{\cdot}873}\int \frac{y^3 e^{-y}dy}{e^y-1}}{\int \frac{1-e^{-y}}{y}dy + \int \tfrac{1}{2}\frac{e^{-y}dy}{y} + \frac{\sqrt{3}}{16}\frac{1}{0{\cdot}873}\int \frac{y^3 e^{-2y}dy}{e^y-1}}.$$

The limits of integration are 0 and ∞. At the zero limit we have the same difficulty as in the previous case, since the first numerator and the second denominator integral are infinite for this limit. Here also the infinity must be removed by cutting off the higher levels at some value of $y = \delta$ (corresponding to $h\nu_1/\omega^2 kT$ in the hydrogen case).

For the upper limit ∞ the first integrals in numerator and denominator both become infinite. This infinity, however, is not real, since it is introduced by our assuming a smoothed formula continually going up for higher ν, whereas the real k, above the last band edge belonging to the lowest level, decreases as ν^{-3}. So we may use the smoothed function as far as ν_1 corre-

* *Publ. Amst.*, IV, p. 32.

sponding to the lowest level, and then from $Y = h\nu_1/kT$ up to ∞ we have to use the value $k_1(\nu_0/\nu)^3$. For the latter domain of integration the integrals are

$$\tfrac{1}{2}k_1\nu_0^3\int_Y^\infty \frac{dy}{y(e^y-1)} = \tfrac{1}{2}k_1\nu_0^3 A(Y); \qquad k_1\nu_0^3\int_Y^\infty \frac{e^{-y}dy}{y} = -k_1\nu_0^3 Ei(-Y);$$

$$\tfrac{1}{2}k_1\nu_0^3\int_Y^\infty \frac{e^{-y}dy}{y(e^y-1)} = \tfrac{1}{2}k_1\nu_0^3\{A(Y)+Ei(-Y)\},$$

where, after division by the factors omitted above,

$$k_1\nu_0^3 = C_1/C_0\{g_1 e^{h\nu_1/kT} + g_2 e^{h\nu_2/kT} + \ldots\}.$$

The last integrals in numerator and denominator represent the influence of temperature increase. Since we have

$$\int_0^\infty \frac{y^3 e^{-y}dy}{e^y-1} = \tfrac{1}{15}\pi^4 - 6 = 0{\cdot}4940 \quad \text{and} \quad \int_0^\infty \frac{y^3 e^{-2y}dy}{e^y-1} = \tfrac{1}{15}\pi^4 - 6\tfrac{3}{8} = 0{\cdot}1190,$$

their numerical values are 0·061 and 0·015. So now the correction factor is

$$\frac{N}{D} = \frac{\tfrac{1}{2}k_1\nu_0^3 \,.\, A(Y) + \tfrac{1}{2}(lnY - ln\delta) + 0{\cdot}061}{\begin{aligned} &-k_1\nu_0^3 Ei(-Y) + lnY + Ei(-Y) - (ln\delta + Ei(-\delta)) + \tfrac{1}{2}k_1\nu_0^3\{A(Y)+Ei(-Y)\} \\ &\qquad -\tfrac{1}{2}Ei(-Y) + \tfrac{1}{2}Ei(-\delta) + 0{\cdot}015, \end{aligned}}$$

where for small values of δ, $ln\delta + Ei(-\delta) = \delta - \gamma$.

For a numerical example we take the case of *Ca* for the solar temperature

$$5040/T = 0{\cdot}87; \qquad C_1/C_0 = [1{\cdot}55]; \qquad V = 6{\cdot}09, \qquad Y = 0{\cdot}87V \times 2{\cdot}30 = 12{\cdot}18,$$
$$k_1\nu_0^3 = [4{\cdot}98], \quad A(Y) = [0{\cdot}583 - 7].$$

For δ we take 0·12 and 0·06. Then we have

N		D_1		D_2	
k_1 term	0·018	k_1 term	0·036	k_1 term	0·000
$\tfrac{1}{2}lnY$	1·250	$lnY + Ei(-Y)$	2·500	$-\tfrac{1}{2}Ei(-Y)$	0·000
$-\tfrac{1}{2}ln\delta$	+1·060 / +1·407	$\gamma - \delta$	+0·457 / +0·517	$\tfrac{1}{2}Ei(-\delta)$	0·832 / 1·179

and $N/D = 2{\cdot}389/3{\cdot}840 = 0{\cdot}62$ and $2{\cdot}736/4{\cdot}247 = 0{\cdot}64$ in the two cases for δ. Here again we see that the result is near to 0·5 when the first terms dominate, but that with increasing importance of the tail, *i.e.* for smaller values of δ, the ratio increases towards 1. The influence of the temperature increase with depth is very small. So qualitatively the results are the same as in the case of hydrogen absorption. And here also we have to consider that the observed intensities of lines are not determined by the ionization of this uppermost layer only but by the ionization in all the deeper relevant layers too.

Excitation in the Upper Layers.—We take the simple case of a second level, which is fed by transitions from the first to the second level. The radiation ν_{12} producing these transitions is almost totally absorbed when reaching the upper layers; its intensity may be denoted by $I = rI(0)$ (r = residual fraction). If in the equilibrium condition, expressed in

Einstein probabilities $n_1 B_{12} I = n_2(A_{21} + B_{21} I)$, we introduce the values of these probabilities we find

$$\frac{n_2}{n_1} = \frac{p_2}{p_1} \frac{I}{2h\nu^3/c^2 + I} = \frac{p_2}{p_1} \frac{r}{e^{h\nu/kT} - 1 + r} \sim \frac{p_2}{p_1} r e^{-h\nu/kT}.$$

Strictly speaking we should divide up the realm of the absorption line ν_{12} in its separate $d\nu$, and take the sum total of the results. Since, however, for small r $1/r \sim \sqrt{B_{12}}$ (because the diffusion coefficient $s \sim B_{12}$ and $r \sim \sqrt{k/(k+s)}$) the central parts with large s and B contribute most to the result, we may restrict ourselves to these central parts and make use of an average r for these parts. Moreover, the fraction of atoms occupying the 2nd level will deviate from this result because of the interaction of this level with the ionized state.

This holds for the uppermost layer. For other layers the intensity (in the case of two vertical radiation streams) is given by *

$$E_0\left(1 + ct - \frac{m - c/m}{1 + m} e^{-mt}\right), \qquad m = \sqrt{(k+s)/k}.$$

Since for the central parts of the line, which chiefly are responsible for lifting the atoms into the second state of energy, s is extremely large, it follows that m is also very large, the exponential term decreases rapidly with t and is only perceptible in the highest layers. For small values already of the optical depth (measured by the mean absorption) the ν_{12} radiation is given by $E_0(1 + ct)$, *i.e.* by black-body radiation of the local temperature. Everywhere in the atmosphere, except in an extreme thin top layer of an optical thickness of $\sqrt{k/s}$, the exciting radiation is black-body radiation, first of temperature T_0 and then increasing downward with increasing temperature. This is obvious, because through the strong absorption coefficient an atom is reached by ν_{12} radiation from the immediate neighbourhood only. Hence, except for that thin surface layer, the number of excited atoms is given for the upper layers by $p_2/p_1 e^{-h\nu_{12}/kT_0}$, and then for deeper layers by $p_2/p_1 \,.\, e^{-h\nu_{12}/kT}$.

The observed intensity of an absorption line depends on the breadth of the profile where the residual intensity r is nearly $\frac{1}{2}$. These parts of the profile are determined by layers with an optical depth of nearly $\frac{1}{3}$, say between 0 and 1. In each layer the excitation depends on the local temperature in this layer. Since at an optical depth $\frac{2}{3}$ the local temperature is the effective temperature itself, we may say that the mean temperature determining the excitation and appearing in the exponent of the Boltzmann function, is somewhat lower than the effective temperature, but higher than the surface temperature. The old formula $\frac{1}{2} \exp(h\nu/kT_1)$, though not correct now as to the surface layers, will give results of the right order of magnitude for a line produced by the total atmosphere.

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* Cf. *M.N.*, 95, 729.

NON-COHERENT DISPERSION AND THE FORMATION OF FRAUNHOFER LINES.

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Abstract.—Weisskopf and Wigner's theory of natural line breadth is applied to the formation of Fraunhofer lines. Formulæ are found for the transfer of radiation from the wings to the centre of a line and *vice versa*. The effect for principal lines, originating in the ground state, is found to be equal in magnitude to that for secondary lines, but less in relative importance. The equation of radiative transfer is suitably modified and its implications briefly discussed.

1. *Introduction.*—The present theories of stellar absorption-line profiles are based almost entirely on what has been called coherent dispersion or scattering. The assumption is made that the emission in a given frequency is dependent only on the absorption of that same frequency, with perhaps some dependence on other lines and on the entire continuous spectrum.

The suggestion that this assumption might not be valid, that actually radiation of frequency ν might be first absorbed, and then subsequently re-emitted with a frequency $\nu+\Delta\nu$, was first made by Eddington.* The subject was examined by Woolley † and Pannekoek,‡ who concluded that if coupling existed between the centre and the wings of a line, it was without much effect on the central intensity. Neither of these investigations was based on a very definite physical theory.

Such a theory has been provided by the work of Weisskopf and Wigner.§ On the basis of Dirac's theory of radiation, they derived general formulæ for the probabilities of cycles of transitions. Similar analyses have been made by Rosenfeld,‖ Casimir ¶ and Hoyt.** The theory has been further refined and generalized by Breit †† and Lowen,‡‡ but the treatment by Weisskopf and Wigner is sufficiently general for our purpose.

The doubt which the experimental results of Shankland have recently cast on quantum electrodynamics in general and on the detailed conservation of energy in particular, may occasion some misgivings as to the validity of the Weisskopf-Wigner treatment. There are two reasons for reassurance on this point. In the first place, the calculation of absorption and emission coefficients is an essentially statistical affair. There is reason to believe that the conclusions of wave mechanics, including the conservation of

* *M.N.*, **89**, 620, 1929.
† *M.N.*, **90**, 779, 1930 ; **91**, 977, 1931 ; *Ap. J.*, **73**, 194, 1931.
‡ *Proc. Roy. Acad. Sciences Amst.*, **34–2**, 1353, 1931.
§ *Zs. f. Phys.*, **63**, 54, 1930 ; **85**, 451, 1933 ; *Ann. d. Phys.*, **9**, 213, 1931.
‖ *Zs. f. Phys.*, **71**, 273, 1931. ¶ *Zs. f. Phys.*, **81**, 506, 1933.
** *Phys. Rev.*, **36**, 860, 1930. †† *Rev. Mod. Ph.*, **4**, 504, 1932 ; **5**, 91, 1933.
‡‡ Breit and Lowen, *Phys. Rev.*, **46**, 590, 1934.

energy, are statistically correct. As long as we do not apply our formulæ to experiments concerning only a few atoms we should be on safe ground.

Secondly, the formulæ which we shall use are consistent with classical theory. Halpern * has shown that if a train of light waves is assumed to be finite in length, transfers of energy will occur from one range of frequencies to another in much the same way as given by the quantum theory.

2. *Theory of Secondary Lines.*—In applying the theory we shall find it advantageous to consider an assembly of highly schematized atoms, in which transitions from one level to another are instantaneous and in which the energy of any electron has a definite meaning. Each state, furthermore, is taken to cover a continuous range of energy levels ; in thermodynamic equilibrium the probability that an electron, known to be in some state P, will also have an energy between E_i and E_i+dE is assumed to be given by

$$\left(\frac{\gamma_P h}{\pi}\right)\frac{dE}{h^2\gamma_P{}^2+(E_i-E_P)^2}, \tag{1}$$

where E_P is the "central" energy of the P level, and

$$\gamma_P=\frac{1}{4\pi}\frac{1}{\tau_P}.$$

Since τ_P is defined as the mean life of an electron in the P level, γ_P corresponds to a range of approximately 10^{-2} angstroms ; γ_P is therefore small in comparison with the observed widths of the lines whose contours we discuss. The factor in parenthesis in (1) is inserted for normalization of the probability.

As a detailed picture of a quantum mechanical state all this is, of course, incorrect. But statistically the model yields exactly the same emission and absorption formulæ as the more complicated treatment, a fact pointed out by Weisskopf and Wigner in their first paper. Because of the greater convenience of this point of view we shall adopt it throughout.

For convenience we first make a change of notation. Let

$$s_P\equiv\frac{E_i-E_P}{h},$$

and (1) now becomes

$$\frac{\gamma_P}{\pi}\frac{ds_P}{\gamma_P{}^2+s_P{}^2}.$$

The transition probabilities are easily derived from this expression. If b_{QP} is the Einstein probability coefficient for a spontaneous downwards transition from level Q to level P, and ds_P, which we shall call an element of P, denotes a range of energy levels between s_P and s_P+ds_P, then the probability per electron per unit time of a transition from any element of Q to an element of P will be given by

$$b_{QP}\frac{\gamma_P}{\pi}\frac{ds_P}{\gamma_P{}^2+s_P{}^2}. \tag{2}$$

* *Zs. f. Phys.*, 67, 523, 1931.

The corresponding probability for a stimulated upwards transition from P to Q depends on $\mathcal{J}(\nu)$, or $c/4\pi$ times the total energy density per unit frequency of the absorbed radiation, and therefore depends on the initial as well as the terminal element. The formula is

$$a_{PQ}\frac{\gamma_Q}{\pi}\frac{ds_Q}{\gamma_Q^2+s_Q^2}\mathcal{J}(\nu_P{}^Q+s_Q-s_P), \tag{3}$$

where

$$\nu_P{}^Q \equiv \frac{E_Q-E_P}{h}.$$

We neglect stimulated downwards transitions.

From (3) it is evident that when $\mathcal{J}(\nu)$ fluctuates over a small range, the distribution of electron energies in each state will not, in general, be that given by (1). To find $n(s_Q)$, defined as the number of electrons per unit range of s_Q at each point, we must solve the equations of statistical equilibrium. We first suppose n_A, the total number of electrons in the ground state A, is known. It is convenient to take n_A as the number of ground state atoms per unit mass of stellar material. In terms of n_A we can then find $n(s_R)$ for all R's, where R is any state.

This calculation rests on the condition that the total number of electrons jumping into an element ds_R (the input) just equals the number of those jumping out (output). It is also assumed that jumps between two elements of the same level do not occur. We can find from (2) and (3) the number of transitions between each element and ds_R, integrate over all elements except ds_R and equate to zero. This gives the equation of equilibrium for ds_R. The set of simultaneous integral equations so obtained may be solved for $n(s_R)$ provided the form of $\mathcal{J}(\nu)$ is known.

Since we are not interested directly in fluorescence, we shall consider an atom with one electron and three states, a ground state A and two upper states B and C, and shall exclude transitions between A and C. This gives two absorption lines—one a principal, the other a secondary line.

Equating the input and output for each element of A, B and C respectively, we find the following three different sets of equations :—

$$b_{BA}\frac{\gamma_A}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_B)}{\gamma_A^2+s_A^2}ds_B = a_{AB}\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_A)\mathcal{J}(\nu_A{}^B+s_B-s_A)}{\gamma_B^2+s_B^2}ds_B, \tag{4}$$

$$\left.\begin{aligned} b_{CB}\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_C)}{\gamma_B^2+s_B^2}ds_C + a_{AB}\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_A)\mathcal{J}(\nu_A{}^B+s_B-s_A)}{\gamma_B^2+s_B^2}ds_A \\ = b_{BA}\frac{\gamma_A}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_B)}{\gamma_A^2+s_A^2}ds_A + a_{BC}\frac{\gamma_C}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_B)\mathcal{J}(\nu_B{}^C+s_C-s_B)}{\gamma_C^2+s_C^2}ds_C \end{aligned}\right\}, \tag{5}$$

$$b_{CB}\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_C)}{\gamma_B^2+s_B^2}ds_B = a_{BC}\frac{\gamma_C}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_B)\mathcal{J}(\nu_B{}^C+s_C-s_B)}{\gamma_C^2+s_C^2}ds_B. \tag{6}$$

As a first approximation in solving these we may neglect the terms in b_{CB} and a_{BC} in equation (5). This is tantamount to assuming that $n(s_B)$ is

determined only by state A; that is to say, transitions from B to C are so much fewer than those from B to A that their effect on $n(s_B)$ is negligible. The relative error introduced by this assumption will be roughly the ratio of the numbers of electrons in states C and B, and will be of the same order of magnitude as the ratio of stimulated to spontaneous downwards transitions.

With this assumption we have, solving (4), (5) and (6),

$$n(s_A)=\frac{b_{BA}}{a_{AB}}\frac{\gamma_A}{\gamma_B}\frac{n_B}{{\gamma_A}^2+{s_A}^2}\left\{\int_{-\infty}^{+\infty}\frac{\mathcal{J}({\nu_A}^B+s_B-s_A)}{{\gamma_B}^2+{s_B}^2}ds_B\right\}^{-1}, \tag{7}$$

$$n(s_B)=\frac{a_{AB}}{b_{BA}}\frac{\gamma_B}{\pi}\frac{1}{{\gamma_B}^2+{s_B}^2}\int_{-\infty}^{+\infty}n(s_A)\mathcal{J}({\nu_A}^B+s_B-s_A)ds_A, \tag{8}$$

$$n(s_C)=\frac{a_{BC}}{b_{CB}}\frac{\gamma_C}{\pi}\frac{1}{{\gamma_C}^2+{s_C}^2}\int_{-\infty}^{+\infty}n(s_B)\mathcal{J}({\nu_B}^C+s_C-s_B)ds_B, \tag{9}$$

where for any state R,

$$n_R=\int_{-\infty}^{+\infty}n(s_R)ds_R.$$

We now assume that the mean life of a ground-state electron is very much greater than the mean life in either of the two excited states, and that therefore γ_A is very small compared with γ_B or γ_C. $\mathcal{J}({\nu_A}^B+s_A)$ may then be considered as a constant over s_A, and the integrals in (7) and (8) may be evaluated by setting s_A equal to zero in the argument of $\mathcal{J}(\nu)$. Let

$$\sigma\equiv\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{\mathcal{J}({\nu_A}^B+s_B)}{{\gamma_B}^2+{s_B}^2}ds_B. \tag{10}$$

Then, since τ_A is the reciprocal of the total probability per unit time per electron in A of the transition A to B,

$$\gamma_A=\frac{1}{4\pi}a_{AB}\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{\mathcal{J}({\nu_A}^B+s_B)}{{\gamma_B}^2+{s_B}^2}ds_B=\frac{1}{4\pi}a_{AB}\sigma. \tag{11}$$

Similarly for states B and C,

$$\gamma_B=\frac{1}{4\pi}b_{BA},\qquad \gamma_C=\frac{1}{4\pi}b_{CB}. \tag{12}$$

These give, substituting in (7), (8) and (9), and integrating (7) for n_A,

$$n_A=\frac{\gamma_B}{\gamma_A}n_B, \tag{13}$$

$$n(s_B)=\frac{\gamma_A}{\pi\sigma}n_A\frac{\mathcal{J}({\nu_A}^B+s_B)}{{\gamma_B}^2+{s_B}^2}, \tag{14}$$

$$n(s_C)=\frac{\gamma_A}{4\pi^3\sigma}\frac{n_A a_{BC}}{{\gamma_C}^2+{s_C}^2}\int_{-\infty}^{+\infty}\frac{\mathcal{J}({\nu_B}^C+s_C-s_B)\,\mathcal{J}({\nu_A}^B+s_B)}{{\gamma_B}^2+{s_B}^2}ds_B. \tag{15}$$

Now the total absorption and emission of radiation with frequency ν are determined from the integrated number of appropriate transitions, multiplied by $h\nu$ to give the energy emitted or absorbed; this, divided by 4π, gives the average per unit solid angle. Let $l(\nu)\mathcal{J}(\nu)$ be the absorption of radiation with frequency ν per unit frequency, per unit mass of stellar matter, where $\mathcal{J}(\nu)$ is as defined above. Let $j(\nu)$ be the corresponding emission per unit solid angle. Then we have, multiplying (3) and (2) by $n(s_B)h\nu/4\pi$ and $n(s_C)h\nu/4\pi$ respectively, and integrating,

$$l(\nu_B{}^C+\Delta\nu)\mathcal{J}(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}a_{BC}\frac{\gamma_C}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_B)\mathcal{J}(\nu_B{}^C+\Delta\nu)}{\gamma_C{}^2+(s_B+\Delta\nu)^2}ds_B, \tag{16}$$

$$j(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}b_{CB}\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_C)}{\gamma_B{}^2+(s_C-\Delta\nu)^2}ds_C. \tag{17}$$

Subscripts and superscripts are omitted from $h\nu$, since the frequency to which ν refers is obvious. The integration is evidently over all s_B and s_C satisfying the relationship,

$$\Delta\nu=s_C-s_B.$$

Substituting (13), (14) and (15) into (16) and (17), we have, respectively,

$$l(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}\frac{a_{BC}\gamma_C\gamma_A n_A}{\pi^2\sigma}\int_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_A{}^B+s_B)}{\{\gamma_B{}^2+s_B{}^2\}\{\gamma_C{}^2+(s_B+\Delta\nu)^2\}}ds_B, \tag{18}$$

$$j(\nu_B{}^C+\Delta\nu)=\left[\frac{h\nu}{4\pi}\frac{b_{CB}\gamma_A\gamma_B a_{BC}n_A}{4\pi^4\sigma}\right]\iint_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_B{}^C+s_C-s_B)\mathcal{J}(\nu_A{}^B+s_B)}{\{\gamma_B{}^2+(s_C-\Delta\nu)^2\}\{\gamma_C{}^2+s_C{}^2\}\{\gamma_B{}^2+s_B{}^2\}}ds_Bds_C.$$

Let

$$s_B\equiv s_C-x.$$

Then, changing the order of integration, and interchanging limits, we have

$$j(\nu_B{}^C+\Delta\nu)=[\text{——}]\int_{-\infty}^{+\infty}\mathcal{J}(\nu_B{}^C+x)dx \int_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_A{}^B+s_C-x)}{\{\gamma_B{}^2+(s_C-\Delta\nu)^2\}\{\gamma_C{}^2+s_C{}^2\}\{\gamma_B{}^2+(s_C-x)^2\}}ds_C. \tag{19}$$

To evaluate (18) and (19) we may avail ourselves of the fact that γ_B and γ_C are usually small, corresponding to less than 10^{-2} angstroms. Although $\mathcal{J}(\nu)$ changes across the observed line width, it will change hardly at all in a frequency interval of the order of γ_B or γ_C. We may therefore find a first approximation to $l(\nu_B{}^C+\Delta\nu)$ and $j(\nu_B{}^C+\Delta\nu)$ by considering $\mathcal{J}(\nu)$ as a constant over each small range of integration whose order of magnitude is several times γ_B or γ_C as the case may be. The validity of this approximation will be discussed in § 4.

Now the integral in (18) is largely made up of those values of s_B which minimize either of the two factors in the denominator, *i.e.* $-\Delta\nu$ and 0. We therefore take $\mathcal{J}(\nu)$ out of the integral for these two values in the argument, and set

$$\int_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_A{}^B+s_B)}{\{\gamma_B{}^2+s_B{}^2\}\{\gamma_C{}^2+(s_B+\Delta\nu)^2\}}ds_B=\frac{\mathcal{J}(\nu_A{}^B)}{\gamma_C{}^2+(\Delta\nu)^2}\int_{-\infty}^{+\infty}\frac{ds_B}{\gamma_B{}^2+s_B{}^2}$$
$$+\frac{\mathcal{J}(\nu_A{}^B-\Delta\nu)}{\gamma_C{}^2+(\Delta\nu)^2}\int_{-\infty}^{+\infty}\frac{ds_B}{\gamma_C{}^2+(s_B+\Delta\nu)^2},$$
$$=\frac{\pi}{\gamma_B\gamma_C}\left\{\frac{\gamma_C\mathcal{J}(\nu_A{}^B)}{\gamma_C{}^2+(\Delta\nu)^2}+\frac{\gamma_B\mathcal{J}(\nu_A{}^B-\Delta\nu)}{\gamma_B{}^2+(\Delta\nu)^2}\right\}.$$

If we let $\mathcal{J}(\nu)$ equal a constant K then the integral can be evaluated exactly, and

$$\int_{-\infty}^{+\infty}\frac{K}{\{\gamma_B{}^2+s_B{}^2\}\{\gamma_C{}^2+(s_B+\Delta\nu)^2\}}ds_B=\frac{\pi}{\gamma_B\gamma_C}\frac{K(\gamma_B+\gamma_C)}{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2}.$$

We see that our evaluation of the integral is valid when $\Delta\nu$ is greater than $\gamma_B+\gamma_C$. To obtain a result accurate for all $\Delta\nu$, we set, substituting from (13) in (18),

$$l(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}\frac{n_Ba_{BC}}{\pi\sigma}\left\{\frac{\gamma_C\mathcal{J}(\nu_A{}^B)+\gamma_B\mathcal{J}(\nu_A{}^B-\Delta\nu)}{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2}\right\}. \tag{20}$$

Similarly for (19), the integrand will be maximized, for large values of $\Delta\nu$, when (*a*) $s_C=0$, $x=0$, and (*b*) $s_C=\Delta\nu$, $x=\Delta\nu$. In both cases s_C-x is equal to zero, and the term $\mathcal{J}(\nu_A{}^B+s_C-x)$ may be taken out of the right-hand integral, leaving an expression which may be evaluated accurately. This gives

$$\int_{-\infty}^{+\infty}\mathcal{J}(\nu_B{}^C+x)dx\int_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_A{}^B+s_C-x)}{\{\gamma_B{}^2+(s_C-\Delta\nu)^2\}\{\gamma_C{}^2+s_C{}^2\}\{\gamma_B{}^2+(s_C-x)^2\}}ds_C$$
$$=\frac{\pi}{\gamma_B\gamma_C}\frac{\mathcal{J}(\nu_A{}^B)}{\{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2\}\{(\gamma_B-\gamma_C)^2+(\Delta\nu)^2\}}.$$

$$\int_{-\infty}^{+\infty}\mathcal{J}(\nu_B{}^C+x)dx\left\{2\gamma_C\frac{(\gamma_C{}^2-\gamma_B{}^2+2(\Delta\nu)^2-x\Delta\nu)}{(2\gamma_B)^2+(x-\Delta\nu)^2}\right.$$
$$\left.+\frac{(\gamma_B+\gamma_C)(\gamma_B{}^2-\gamma_C{}^2+(\Delta\nu)^2)+2\gamma_Cx\Delta\nu}{(\gamma_B+\gamma_C)^2+x^2}\right\}.$$

Repeating the same process on $\mathcal{J}(\nu_B{}^C+x)$, this expression reduces to

$$\frac{\pi^2}{\gamma_B{}^2\gamma_C}\mathcal{J}(\nu_A{}^B)\left\{\frac{\gamma_C\mathcal{J}(\nu_B{}^C+\Delta\nu)(\gamma_C{}^2-\gamma_B{}^2+(\Delta\nu)^2)+\gamma_B\mathcal{J}(\nu_B{}^C)(\gamma_B{}^2-\gamma_C{}^2+(\Delta\nu)^2)}{\{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2\}\{(\gamma_B-\gamma_C)^2+(\Delta\nu)^2\}}\right\}.$$

For $\Delta\nu$ greater than γ_B or γ_C, we have, substituting from (12) and (14) in (19),

$$j(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}\frac{n_Ba_{BC}}{\pi}\frac{\mathcal{J}(\nu_A{}^B)}{\sigma}\left\{\frac{\gamma_C\mathcal{J}(\nu_B{}^C+\Delta\nu)+\gamma_B\mathcal{J}(\nu_B{}^C)}{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2}\right\}. \tag{21}$$

Since the absorption minus the emission is the significant term in the equation of radiative transfer, we define $L(\nu)$ as

$$L(\nu)\equiv l(\nu)\mathcal{J}(\nu)-j(\nu).$$

Combining (20) and (21) we now have

$$L(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B+\gamma_C}{\pi}\frac{n_B a_{BC}}{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2}\frac{\gamma_B}{\gamma_B+\gamma_C}$$
$$\left\{\frac{\mathcal{J}(\nu_B{}^C+\Delta\nu)\mathcal{J}(\nu_A{}^B-\Delta\nu)-\mathcal{J}(\nu_B{}^C)\mathcal{J}(\nu_A{}^B)}{\sigma}\right\}. \quad (22)$$

When $\Delta\nu$ is small, $L(\nu_B{}^C+\Delta\nu)$ as given by (22) vanishes. To derive a better approximation in this case we may combine (18) and (19) directly to find an integral expression for $L(\nu_B{}^C+\Delta\nu)$. If we let s_B equal $s_C-\Delta\nu$ in (18), and x equal s_C-w in (19), integrating over dw rather than dx, we find

$$L(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}\frac{a_{BC}n_A}{\sigma}\frac{\gamma_A\gamma_B\gamma_C}{\pi^3}$$
$$\iint_{-\infty}^{+\infty}\left\{\frac{\mathcal{J}(\nu_A{}^B+s_C-\Delta\nu)\mathcal{J}(\nu_B{}^C+\Delta\nu)-\mathcal{J}(\nu_A{}^B+w)\mathcal{J}(\nu_B{}^C-w+s_C)}{\{\gamma_B{}^2+w^2\}\{\gamma_C{}^2+s_C{}^2\}\{\gamma_B{}^2+(s_C-\Delta\nu)^2\}}\right\}ds_C dw. \quad (23)$$

When $\Delta\nu$ is large the integral will be maximized at $w=0$, $s_C=0$; (23) then reduces to (22). When $\Delta\nu$ is sufficiently small so that $\mathcal{J}(\nu)=\mathcal{J}(\nu+\Delta\nu)$ for all ν, then to the same order of approximation as before, the integrand is maximized for s_C small. We therefore set s_C equal to zero in the numerator and find, substituting for n_A from (13),

$$L(\nu_B{}^C+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B+\gamma_C}{\pi}\frac{n_B a_{BC}}{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2}\frac{\gamma_B}{\pi\sigma}$$
$$\int_{-\infty}^{+\infty}\left\{\frac{\mathcal{J}(\nu_B{}^C)\mathcal{J}(\nu_A{}^B)-\mathcal{J}(\nu_B{}^C+w)\mathcal{J}(\nu_A{}^B-w)}{\gamma_B{}^2+w^2}\right\}dw. \quad (24)$$

In formulæ (20) to (24) σ may be taken equal to $\mathcal{J}(\nu_A{}^B)$ from (10). If a more accurate expression is desired, we have from (11), (12) and (13),

$$\frac{n_B}{\sigma}=n_A\frac{a_{AB}}{b_{BA}}=n_A\frac{q_B}{q_A}\frac{c^2}{2h(\nu_A{}^B)^3}. \quad (25)$$

3. *Theory of Principal Lines.*—We now consider the effects on $j(\nu_A{}^B+\Delta\nu)$ of the second approximation to $n(s_B)$. This is necessary, since to a first approximation $L(\nu_A{}^B+\Delta\nu)$ vanishes.

With the usual approximations, we have from (5)

$$n(s_B)=\frac{\gamma_B}{\pi}\frac{1}{\gamma_B{}^2+s_B{}^2}\left\{\frac{n_C b_{CB}+n_A a_{AB}\mathcal{J}(\nu_A{}^B+s_B)}{b_{BA}+a_{BC}\mathcal{J}(\nu_B{}^C-s_B)}\right\},$$

which, since $a_{BC}\mathcal{J}(\nu_B{}^C-s_B)/b_{BA}$ is very small compared to unity, reduces to

$$n(s_B)=\frac{\gamma_B}{\pi}\frac{1}{\gamma_B{}^2+s_B{}^2}\frac{1}{b_{BA}}(n_C b_{CB}+n_A a_{AB}\mathcal{J}(\nu_A{}^B+s_B))\left(1-\frac{a_{BC}}{b_{BA}}\mathcal{J}(\nu_B{}^C-s_B)\right). \quad (26)$$

Now the emission is given by

$$j(\nu_A{}^B+\Delta\nu)=\frac{h\nu}{4\pi}b_{BA}\frac{\gamma_A}{\pi}\int_{-\infty}^{+\infty}\frac{n(s_B)}{\gamma_A{}^2+(s_B-\Delta\nu)^2}ds_B,$$
$$=\frac{h\nu}{4\pi}b_{BA}n(\Delta\nu_B),$$

where $n(\Delta\nu_B)$ denotes $n(s_B)$ with s_B equal to $\Delta\nu$.

From the A to B analogue of (16) we have

$$l(\nu_A{}^B+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B}{\pi}\frac{n_Aa_{AB}}{\gamma_B{}^2+(\Delta\nu)^2}, \tag{27}$$

which gives, neglecting squares of second-order terms,

$$L(\nu_A{}^B+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B}{\pi}\frac{1}{\gamma_B{}^2+(\Delta\nu)^2}\left\{\frac{n_Aa_{AB}a_{BC}}{b_{BA}}\mathcal{J}(\nu_A{}^B+\Delta\nu)\mathcal{J}(\nu_B{}^C-\Delta\nu)-n_Cb_{CB}\right\}.$$

Solving (15) for n_C, approximating as before, and substituting above, we have

$$L(\nu_A{}^B+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B}{\pi}\frac{n_Aa_{AB}}{\gamma_B{}^2+(\Delta\nu)^2}\frac{a_{BC}}{b_{BA}}\left\{\mathcal{J}(\nu_A{}^B+\Delta\nu)\mathcal{J}(\nu_B{}^C-\Delta\nu)-\mathcal{J}(\nu_A{}^B)\mathcal{J}(\nu_B{}^C)\right\}. \tag{28}$$

For $\Delta\nu$ small our approximation again breaks down, and more accurately we find

$$L(\nu_A{}^B+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B}{\pi}\frac{n_Aa_{AB}}{\gamma_B{}^2+(\Delta\nu)^2}\frac{a_{BC}\gamma_C}{4\pi^3}$$
$$\iint_{-\infty}^{+\infty}\left\{\frac{\mathcal{J}(\nu_A{}^B+\Delta\nu)\mathcal{J}(\nu_B{}^C+s_C-\Delta\nu)-\mathcal{J}(\nu_A{}^B+s_B)\mathcal{J}(\nu_B{}^C+s_C-s_B)}{(\gamma_B{}^2+s_B{}^2)(\gamma_C{}^2+s_C{}^2)}\right\}ds_Bds_C,$$

where the only approximations are the assumption that γ_A is very small compared to the range over which $\mathcal{J}(\nu_A{}^B+\Delta\nu)$ varies, and the neglect in the denominator of a term

$$1+\frac{a_{BC}}{b_{BA}}\frac{\gamma_C}{\pi}\int_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_B{}^C+s_C-\Delta\nu)}{\gamma_C{}^2+s_C{}^2}ds_C.$$

Now if $\Delta\nu$ is large we again find (28). When $\Delta\nu$ is small the numerator vanishes for s_B equal to zero, whatever the value of s_C. We therefore have

$$L(\nu_A{}^B+\Delta\nu)=\frac{h\nu}{4\pi}\frac{\gamma_B}{\pi}\frac{n_Aa_{AB}}{\gamma_B{}^2+(\Delta\nu)^2}\frac{a_{BC}}{b_{BA}}$$
$$\frac{\gamma_B}{\pi}\int_{-\infty}^{+\infty}\left\{\frac{\mathcal{J}(\nu_A{}^B)\mathcal{J}(\nu_B{}^C)-\mathcal{J}(\nu_A{}^B+s_B)\mathcal{J}(\nu_B{}^C-s_B)}{\gamma_B{}^2+s_B{}^2}\right\}ds_B. \tag{29}$$

Note that from (25) we have

$$\frac{\gamma_B}{\pi}n_Aa_{AB}\frac{a_{BC}}{b_{BA}}=\frac{\gamma_B+\gamma_C}{\pi}n_Ba_{BC}\frac{1}{\sigma}\frac{\gamma_B}{\gamma_B+\gamma_C},$$

and therefore for large $\Delta\nu$,

$$L(\nu_B{}^C + \Delta\nu) = L(\nu_A{}^B - \Delta\nu).$$

For small $\Delta\nu$ the two quantities are exactly equal only if γ_C is zero, although the integrals of $L(\nu)$ over the line centres will be equal in any case. For both the upper and lower line, furthermore, the integral of $L(\nu)$ over the entire

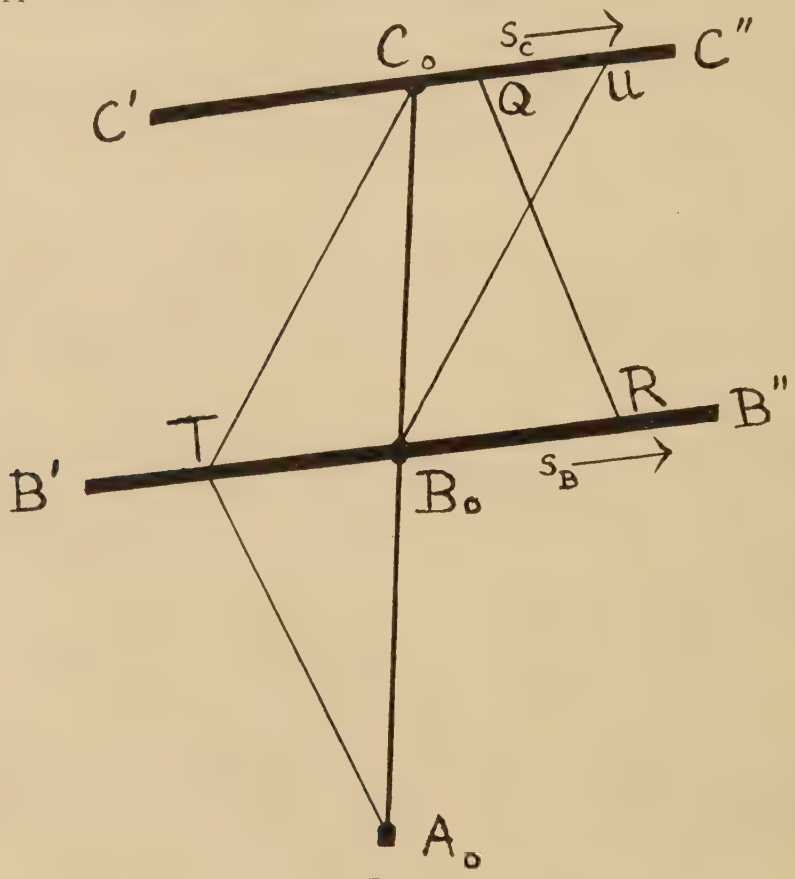

FIG. 1.

line vanishes. This is to be expected, and demonstrates the consistency of our approximations.

4. *Diagrammatic Representation.*—These analyses will be clarified by reference to the accompanying diagram (fig. 1). The three heavy lines A_0, $B'B''$, $C'C''$ represent the three states A, B and C. The length of each of these three lines is proportional to γ_A, γ_B and γ_C respectively. A_0, B_0 and C_0 represent the central energies of each level. Energies greater than the central energy, B_0 say, are denoted on the $B'B''$ line by a point to the right of B_0, those less by a point to the left. The distance of a point from B_0 is taken as proportional to s_B. The line A is so short that we regard it as a single point A_0.

On this scheme a transition between two electron levels will be repre-

sented by a line drawn between the points representing the levels. If $(s_B)_R$ denotes the s_B distance of the point R, any transition R to Q represents a frequency $\nu_R{}^Q$, where

$$\nu_R{}^Q = \nu_B{}^C + (s_C)_Q - (s_B)_R.$$

The length of the line RQ is of no significance.

Now if we assume that the most probable transitions are those with B_0 or C_0 as end points, most of the results in this paper are seen to follow at once. The zero order approximation gives the transitions between B_0 and C_0, the first order, transitions from B_0 or C_0 to some other part of the lines. It is clear that there are only two such lines for a given frequency, $\nu_B{}^C + x$, namely B_0U and TC_0, where

$$(s_C)_U = -(s_B)_T = x.$$

The first provides an absorption coefficient dependent only on n_A and $\mathcal{J}(\nu_A{}^B)$, and an emission equal to the absorption, since the most probable transition for an electron at U is back to B_0.

The second transition TC_0, however, provides an absorption coefficient which is a function of $\mathcal{J}(\nu_A{}^B - x)$ and an emission dependent only on $\mathcal{J}(\nu_A{}^B)$ and $\mathcal{J}(\nu_B{}^C)$, since these determine the number of electrons at C_0. The relative importance of the first transition compared to the second is given by the ratio of $C'C''$ to $B'B''$—in other words, the ratio of γ_C to γ_B.

Similarly, the net effect on $\mathcal{J}(\nu_B{}^C)$ and neighbouring frequencies will be the sum of all cycles $A_0B_0C_0TA_0$, less the sum of all such reverse cycles. We also see that for every shift of radiation frequency in the upper line there must be a corresponding, equal and opposite shift in the lower line. For if we have an unbalanced cycle B_0C_0T transferring radiation from $\mathcal{J}(\nu_B{}^C)$ to $\mathcal{J}(\nu_B{}^C + x)$, the fact that the number of electrons at T must be constant requires that we also have an unbalanced cycle TA_0B_0, transferring an equal amount of radiation from $\mathcal{J}(\nu_A{}^B)$ to $\mathcal{J}(\nu_A{}^B - x)$.

As to the errors introduced by mathematical approximations, they are of two sorts. We have first the assumption that $\mathcal{J}(\nu)$ is sensibly constant over a range of frequencies several times γ_B or γ_C. The corresponding assumption for γ_A is of course quite valid. To evaluate the error introduced by this method of approximation, we first notice that, since all other functions in our integrands are even functions, any linear variation of the product $\mathcal{J}(\nu_B{}^C + \Delta\nu)\mathcal{J}(\nu_A{}^B + s_B - \Delta\nu)$ with $\Delta\nu$ will cancel out on integration. We may then find an upper limit to the error by considering $\mathcal{J}(\nu_B{}^C + \Delta\nu)\mathcal{J}(\nu_A{}^B + s_B - \Delta\nu)$ linear to within a 40 per cent. relative error in an interval $\pm d$ about a given $\nu_B{}^C + \Delta\nu$ (taking s_B as fixed), and equal to zero or $B(\nu_B{}^C, T)B(\nu_A{}^B, T)$ outside the interval, whichever gives the larger error. The relative error in this case is found to be less than

$$\frac{2}{\pi}\frac{\gamma_B + \gamma_C}{d}\frac{B(\nu_B{}^C, T)B(\nu_A{}^B, T)}{\mathcal{J}(\nu_B{}^C + \Delta\nu)\mathcal{J}(\nu_A{}^B + s_B - \Delta\nu)},$$

assuming that the second derivative of $\mathcal{J}(\nu)$ is constant in each half of the interval. $B(\nu, T)$ is of course the intensity of black-body radiation at temperature T.

One would expect d to be much greater than $\gamma_B+\gamma_C$, and therefore the relative error small for appreciable $\mathcal{J}$'s. An exception to this would be the case when the line is so narrow that the Doppler shift becomes predominant and the dispersion becomes coherent in any event.

The other approximation lies in the neglect of stimulated downwards transitions, together with the assumption that we may calculate $l(\nu_B{}^C+\Delta\nu)$ and $j(\nu_B{}^C+\Delta\nu)$ as though $n(s_B)$ were itself unaffected by the transitions B to C to B. When λT is greater than $4\cdot10^7$ (λ in angstroms, T in degrees) this definitely breaks down. As long as we rule out low frequency lines in the hottest stars, however, we should expect our formulæ for $L(\nu)$ to be accurate with a total error of not more than 10 per cent.

5. *Radiative Transfer.*—The equation of radiative transfer is, in the usual notation,*

$$\cos\theta\frac{d\mathcal{J}(\nu,\theta)}{\rho dx}=-(k(\nu)+l(\nu))\mathcal{J}(\nu,\theta)+j(\nu),$$

where $\mathcal{J}(\nu,\theta)$ is the intensity of radiation ν in a direction θ per unit frequency per unit solid angle, and where $k(\nu)$ is the coefficient of continuous absorption. From this one derives

$$\frac{d^2\mathcal{J}(\nu)}{d\tau^2}=3(1+\eta(\nu))\{(1+\eta(\nu))\mathcal{J}(\nu)-j(\nu)\},$$
$$=3\frac{(1+\eta(\nu))}{k(\nu)}L(\nu),$$

where $\eta(\nu)$, the ratio of $l(\nu)$ to $k(\nu)$, is assumed constant, and

$$d\tau\equiv-k(\nu)\rho dx.$$

We must include in $L(\nu)$ the effect of continuous absorption and of inelastic collisions and electron captures. The first merely leads to the addition of a term $\mathcal{J}(\nu)-B(\nu,T)$. The second requires not only the addition of a term $-\epsilon_2\eta^0(\nu)B(\nu,T)$, where the superscript zero denotes the value in thermodynamic equilibrium at the local temperature, but also reduces by a factor $1-\epsilon$ the $j(\nu)$ already derived. We must therefore revise our formulæ for $L(\nu)$.

From (20), (21) and (25) we have, for large $\Delta\nu$,

$$L(\nu_B{}^C+\Delta\nu)=l'(\nu_B{}^C+\Delta\nu)\Bigg\{(1-\epsilon_2)\frac{\gamma_B}{\gamma_B+\gamma_C}\left(\frac{\mathcal{J}(\nu_B{}^C+\Delta\nu)\mathcal{J}(\nu_A{}^B-\Delta\nu)-\mathcal{J}(\nu_B{}^C)\mathcal{J}(\nu_A{}^B)}{\mathcal{J}(\nu_A{}^B)}\right)$$
$$+\epsilon_2\mathcal{J}(\nu_B{}^C+\Delta\nu)+\epsilon_2\frac{\gamma_B}{\gamma_B+\gamma_C}\left(\frac{\mathcal{J}(\nu_A{}^B-\Delta\nu)-\mathcal{J}(\nu_A{}^B)}{\mathcal{J}(\nu_A{}^B)}\right)\mathcal{J}(\nu_B{}^C+\Delta\nu)\Bigg\},\qquad(30)$$

where $l'(\nu_B{}^C+\Delta\nu)$ is by definition

$$l'(\nu_B{}^C+\Delta\nu)\equiv\frac{h\nu}{4\pi}\frac{\gamma_B+\gamma_C}{\pi}\frac{n_Aa_{BC}}{(\gamma_B+\gamma_C)^2+(\Delta\nu)^2}\frac{a_{AB}}{b_{BA}}\mathcal{J}(\nu_A{}^B),$$
$$=\frac{\mathcal{J}(\nu_A{}^B)}{B(\nu_A{}^B,T)}l^0(\nu_B{}^C+\Delta\nu).$$

* Eddington, *loc. cit.*

For $\Delta\nu$ small we find, from (20), (24) and (25),

$$L(\nu_B{}^C+\Delta\nu)=l'(\nu_B{}^C+\Delta\nu)\Big\{(1-\epsilon_2)\frac{\gamma_B}{\pi\mathcal{J}(\nu_A{}^B)}\int_{-\infty}^{+\infty}\frac{\mathcal{J}(\nu_B{}^C)\mathcal{J}(\nu_A{}^B)-\mathcal{J}(\nu_B{}^C+w)\mathcal{J}(\nu_A{}^B-w)}{\gamma_B{}^2+w^2}dw+\epsilon_2\mathcal{J}(\nu_B{}^C+\Delta\nu)\Big\}. \quad (31)$$

It is evident that $L(\nu_A{}^B+\Delta\nu)$ undergoes similar modification, with ϵ_1 replacing ϵ_2, and the last term in (30) dropping out. Now this last term is always less than the second, and is very small when ϵ_1 is appreciable, for the lower line is then relatively undisturbed by coupling with the upper one, and for any $\Delta\nu$ within the upper line, $\mathcal{J}(\nu_A{}^B+\Delta\nu)$ will remain substantially equal to $B(\nu_A{}^B, T)$ until very near the boundary of the atmosphere. We therefore neglect this term.

If, now, we set ν_2 equal to $\nu_B{}^C+\Delta\nu$ for large $\Delta\nu$, and ν_1 equal to the corresponding $\nu_A{}^B-\Delta\nu$, defining ν_4 and ν_3 similarly for small $\Delta\nu$, our four equations of radiative transfer become

$$\frac{d^2\mathcal{J}(\nu_1)}{d\tau^2}=3(1+\eta(\nu_1))\Big\{(1+\epsilon_1\eta(\nu_1))(\mathcal{J}(\nu_1)-B(\nu_1, T))+\beta_1\eta(\nu_1)(\mathcal{J}(\nu_1)\mathcal{J}(\nu_2)-\mathcal{J}(\nu_A{}^B)\mathcal{J}(\nu_B{}^C))\Big\}, \quad (32)$$

$$\frac{d^2\mathcal{J}(\nu_2)}{d\tau^2}=3(1+\eta(\nu_2))\Big\{(1+\epsilon_2\eta'(\nu_2))\mathcal{J}(\nu_2)-(1+\epsilon_2\eta^0(\nu_2))B(\nu_2, T)+\beta_2\eta'(\nu_2)\Big(\frac{\mathcal{J}(\nu_2)\mathcal{J}(\nu_1)-\mathcal{J}(\nu_B{}^C)\mathcal{J}(\nu_A{}^B)}{\mathcal{J}(\nu_A{}^B)}\Big)\Big\}, \quad (33)$$

$$\frac{d^2\mathcal{J}(\nu_3)}{d\tau^2}=3(1+\eta(\nu_3))\Big\{(1+\epsilon_1\eta(\nu_3))(\mathcal{J}(\nu_3)-B(\nu_3, T))+\frac{\gamma_B}{\pi}\beta_1\eta(\nu_3)\int_{-\infty}^{+\infty}\Big(\frac{\mathcal{J}(\nu_A{}^B)\mathcal{J}(\nu_B{}^C)-\mathcal{J}(\nu_A{}^B+x)\mathcal{J}(\nu_B{}^C-x)}{\gamma_B{}^2+x^2}\Big)dx\Big\}, \quad (34)$$

$$\frac{d^2\mathcal{J}(\nu_4)}{d\tau^2}=3(1+\eta(\nu_4))\Big\{(1+\epsilon_2\eta'(\nu_4))\mathcal{J}(\nu_4)-(1+\epsilon_2\eta^0(\nu_4))B(\nu_4, T)+\frac{\gamma_B+\gamma_C}{\pi}\beta_2\frac{\eta'(\nu_4)}{\mathcal{J}(\nu_A{}^B)}\int_{-\infty}^{+\infty}\Big(\frac{\mathcal{J}(\nu_B{}^C)\mathcal{J}(\nu_A{}^B)-\mathcal{J}(\nu_B{}^C+x)\mathcal{J}(\nu_A{}^B-x)}{\gamma_B{}^2+x^2}\Big)dx\Big\}; \quad (35)$$

where

$$\beta_1\equiv(1-\epsilon_1)\frac{a_{BC}}{b_{BA}}, \qquad \beta_2\equiv(1-\epsilon_2)\frac{\gamma_B}{\gamma_B+\gamma_C},$$

$$\eta'(\nu)\equiv\frac{l'(\nu)}{k(\nu)}, \qquad \eta^0(\nu)\equiv\frac{l^0(\nu)}{k(\nu)},$$

$$\eta(\nu)\equiv\frac{l(\nu)}{k(\nu)},$$

and $l(\nu)$ is given by (20) and (27).

Now if we neglect the coupling terms, and assume that $\eta'(\nu)$ and $\eta(\nu)$ are both equal to $\eta^0(\nu)$, *i.e.* that $\mathcal{J}(\nu_A{}^B)$ is equal to $B(\nu_A{}^B, T)$, then the usual solution for $\mathcal{J}(\nu_B{}^C+\Delta\nu)$ is, assuming $\eta(\nu)$ constant and $B(\nu, T)$ linear in τ,

$$\mathcal{J}(\nu)=B(\nu, T)-\frac{\eta(\nu)B(\nu, T_0)e^{-q(\nu)\tau}}{1+\eta(\nu)+\frac{2}{3}q(\nu)},$$

where

$$q^2(\nu) \equiv 3(1+\eta(\nu))(1+\epsilon_2\eta(\nu)).$$

For τ greater than the reciprocal of $q(\nu_B{}^C)$, the intensity for each $\mathcal{J}(\nu)$—the line profile—will be as shown in fig. 2. The centre of the line recovers its equilibrium value much more quickly as we go down into a star than does the rest of the line with a lower $q(\nu)$.

The introduction of coupling will tend to shift radiation from the high central peak, as shown by the arrows. Unless there is some source of

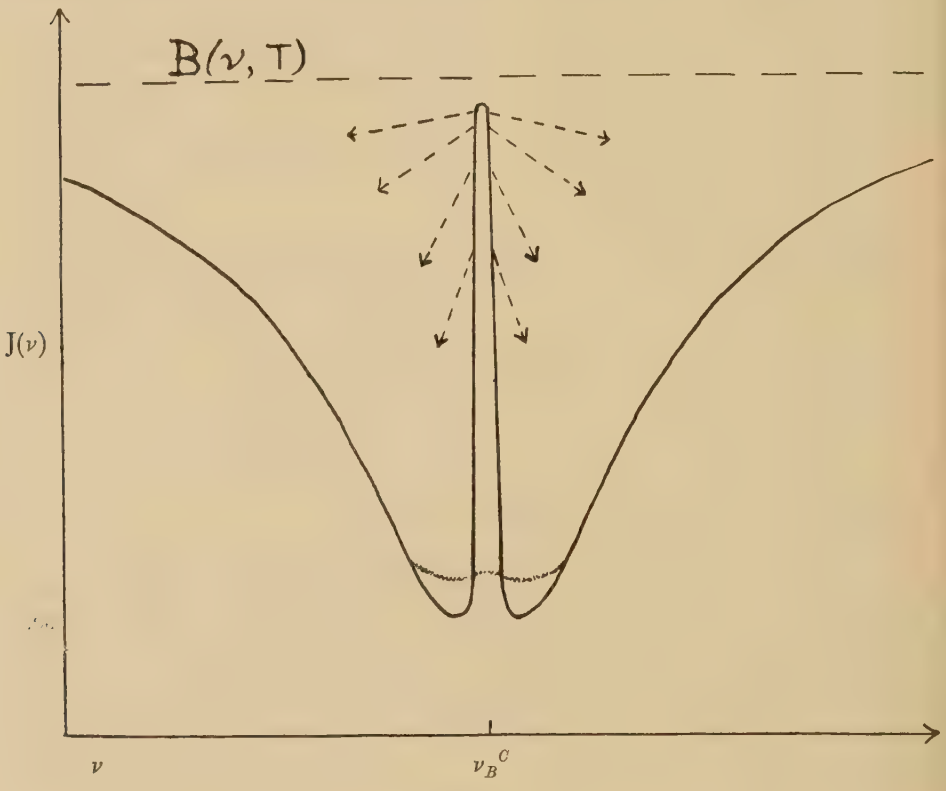

FIG. 2.

radiation to replenish the peak, such as an appreciable ϵ_2, the intensity of the peak will fall to approximately the heavy dotted line in the diagram. In the case when ϵ_2 is negligible, however, and in particular when ϵ_1 is also very small, the phenomena become exceedingly complex. The shift of radiation from the centre to the wings of the upper line will also shift radiation from the centre to the corresponding wings of the lower line. If the lower line is strong, this might even raise $\mathcal{J}(\nu)$ above $B(\nu, T)$ in parts of the lower line.

This case has been considered by Pannekoek,* but neglecting the effect of the lower line. He deals primarily with the central intensity, finding

* *Loc. cit.*

it increased only from a thousandth to a hundredth of the continuous spectrum.

There are two reasons to believe that the case for an appreciable ϵ is the significant one. In the first place, Strömgren * has pointed out the influence of ionization and electron captures in accounting for ϵ. Then the influence of inelastic collisions, although unimportant for central intensities, since these form in high regions of low pressure, is probably of considerable magnitude deeper in the photosphere, where the centre-wings coupling gets in its work.

When ϵ_1 and ϵ_2 are appreciable, the equations simplify considerably. $\mathcal{J}(\nu)$ in the lower line will not be sensitive to coupling with the upper line, as a small change in $\mathcal{J}(\nu_A{}^B + \Delta\nu)$ from its equilibrium value will produce a large "restoring force" $-\epsilon_1\eta(\nu_B{}^A + \Delta\nu)\{\mathcal{J}(\nu_A{}^B + \Delta\nu) - B(\nu_A{}^B + \Delta\nu, T)\}$. There is also the advantage that, since $\mathcal{J}(\nu_A{}^B)$ will retain its equilibrium value until very near the surface of the star, $\eta(\nu_B{}^C + \Delta\nu)$, $\eta'(\nu_B{}^C + \Delta\nu)$ and $\eta^0(\nu_B{}^C + \Delta\nu)$ will all be equal.

In the upper line, which in this case can be considered independently of the lower, $\mathcal{J}(\nu_B{}^C)$ will decrease only slightly because of the coupling drain, as the extra input will be proportional to $\epsilon_2\eta(\nu_B{}^C)\{B(\nu_B{}^C, T) - \mathcal{J}(\nu_B{}^C)\}$. More accurately, a solution of (33) and (35) by successive approximations shows that if $\epsilon_2\eta^{\frac{1}{2}}(\nu_B{}^C)$ is greater than $5{\cdot}10^2$, $\mathcal{J}(\nu_B{}^C)$ will remain within 20 per cent. of its peak value until τ becomes less than 10^{-2}. The quantity $\eta(\nu_B{}^C)$ is the calculated and not the measured ratio of coefficients at the centre of the line. A slight Doppler effect will of course reduce the measured $\eta(\nu_B{}^C)$ to a considerably smaller figure, but as long as the average Doppler shift is less than the width of the central peak for τ greater than ·01, the net effect of coupling on the wings will be the same in general as though no Doppler effect were present. This of course depends on the fact that the wings are fully formed at τ equal to ·01, and provided that $\eta(\nu_B{}^C + \Delta\nu)$ does not increas , they will be quite independent of what happens in the thin layer from τ equal to zero to τ equal to ·01.

In this case Woolley's † equations, which neglect the effect of coupling on the centre itself, are valid for the wings of the line. The net effect is to "amplify" the influence of the ϵ_2, so to speak, and to fill up the wings to an extent depending on the values chosen for γ_B and γ_C. Thus the effect of coupling is very considerable in the wings, although its effect on the central intensity may be negligible.

More detailed discussion of equations (32) to (35), with perhaps applications to specific line contours, will, I hope, be submitted for publication soon.

I am particularly grateful to Professor R. H. Fowler and Professor Sir Arthur Eddington for many helpful and clarifying discussions in the course of these investigations.

St. John's College,
Cambridge :
1936 *May* 25.

* *Zs. f. Ap.*, **10**, 237, 1935.
† *M.N.*, **90**, 779, 1930.

OCCULTATIONS OBSERVED AT THE ROYAL OBSERVATORY, CAPE OF GOOD HOPE, 1932–1935.

(Communicated by H.M. Astronomer)

The occultations observed during the years 1923–1931 were discussed by Dr. H. Spencer Jones and Mr. A. J. Wilkin in *M.N.*, **92**, 846. At the Cape Observatory an attempt is made to observe all the occultations predicted in the *Nautical Almanac*, and only occasionally are the occultations of other stars observed.

In the table given at the end of this paper most of the columns are sufficiently explained by the headings. In the fifth column the first letter D. or R. denotes disappearance or reappearance, the second letter D. or B. whether the phenomenon took place at the dark or bright limb of the Moon. The adopted co-ordinates of the instruments used were as follows :—

Instrument	Longitude	Latitude	Log $\rho \cos \phi'$	Log $\rho \sin \phi'$
	h m s	° ′ ″		
10-inch equatorial	1 13 54·80 E.	−33 56 7·8	9·91936	9·74436*n*
6- ,, ,,	54·74	7·0	6	6
7- ,, heliometer	54·75	1·4	7	4

These data are based on the following position of the 8-inch transit circle :—

Longitude	$1^h\ 13^m\ 54^s{\cdot}70$ E.
Latitude	$-33°\ 56'\ 3''{\cdot}4$
ϕ'	$-33°\ 45'\ 19''{\cdot}8$
Log ρ	9·9995467.

The longitude observations made in 1926 and 1933 indicate that the east longitude should be $0^s{\cdot}10$ smaller, but no change has been made on this account, and the observations have been reduced on the same basis as those published previously.

The seventh column contains the initials of the observers as follows :—

Mr. H. Horrocks	H. H.	Dr. R. H. Stoy	S.
Dr. J. Jackson	J.	Mr. A. J. Wilkin	A. W.
Miss C. Orpen	C. O.		

The column headed U.T. gives the Greenwich Mean Time reckoned from midnight.

The apparent positions of the stars have been taken from the *N.A.* as far as possible ; in other cases the source is given at the end of the table. The positions of the Moon have been interpolated from those given in the *N.A.*, based on Brown's tables of the Moon.

As is well known, the largest part of the error in Brown's tables can be removed by a change in the mean longitude. In recent years a correction of $+5''.0$ has been generally applied, although it is known that the error is gradually decreasing. In the reductions of the Cape observations this correction was applied for the years 1932, 1933 and 1935. For the year 1934 a correction $+4''.4$ was used, but for the sake of uniformity the additional $+0''.6$ has been applied to the true longitude in the resulting equations of condition. All the figures appearing in this paper can be taken as based on a correction of $+5''.0$ to the mean longitude of Brown's tables.

Notes on some of the observations are given at the end of the table.

The reductions have been made by the method introduced by Innes (see *U.O.C.* 55, 56 and 63), which is now generally used for reducing occultations. If σ be the Moon's tabular semi-diameter, σ' the computed distance of the star from the centre of the Moon, each observation gives an equation of the form

$$\begin{aligned}\sigma' - \sigma &= \cos\delta \sin\chi\Delta\alpha + \cos\chi\Delta\delta \\ &= \cos(\chi-\rho)\Delta\lambda - \sin(\chi-\rho)\Delta\beta,\end{aligned}$$

where $\Delta\alpha$ and $\Delta\delta$ or $\Delta\lambda$ and $\Delta\beta$ are the corrections to the adjusted tabular places of the Moon relative to similar corrections to the adopted position of the star. χ is the position angle of the star relative to the Moon's centre at the time of observation, and ρ the position angle of the Moon's motion. When an occultation has been observed by more than one observer the means have been taken in the reductions.

Assuming $\Delta\lambda$ and $\Delta\beta$ constant throughout each year, we find the following corrections to Brown's tables :—

Year	No. of Obs.	Mean Epoch	$\Delta\lambda$	Wt.	$\Delta\beta$	Wt.
			″		″	
1932	20	1932·38	+4·91	12·4	+0·35	5·9
1933	30	33·53	+4·34	20·8	−0·60	9·2
1934	43	34·49	+3·78	30·1	−0·29	12·9
1935	62	35·49	+3·65	35·4	−0·72	27·1

The weighted mean value for $\Delta\beta$ for the four years is $-0''.48$. If we include the determination for the years 1923–1931 we get from all the occultations observed since the introduction of Brown's tables $\Delta\beta = -0''.37$.

Using this value of $\Delta\beta$ we derive from each year's observations the values of $\Delta\lambda$ given in the following table. For comparison the results of the Greenwich meridian observations are given—mean of Limb and Mösting A.

	$\Delta\lambda$	p.e.	Gr.		$\Delta\lambda$	p.e.	Gr.
	″	″	″		″	″	″
1923·67	+7·47	±0·31	+7·43	1930·51	+5·59	±0·27	+5·24
24·48	7·67	·21	7·18	31·27	5·38	·19	4·81
25·64	7·07	·21	6·66	32·38	5·05	·25	4·59
26·20	7·65	·44	6·47	33·53	4·35	·19	4·16
27·20	6·91	·36	6·22	34·49	3·78	·16	3·66
28·36	6·35	·19	5·69	35·49	3·67	·15	
29·50	5·04	·34	5·54				

For the first nine years the probable error of a single observation, used in the calculation of the above probable errors, was ± 0″·79. For the last four years it was ± 0″·88, derived from the residuals using a smoothed value of $\Delta\lambda$ (linear in each year) and $\Delta\beta = -0''{\cdot}37$. This error arises to only a very small extent in the observation of the occultation. It must be largely due to irregularities of the Moon's limb, and to a lesser extent to errors in the adopted star places and in the lunary theory. It should soon be possible from a combination of all occultations observed since 1923 to correlate the residuals with position angle and the phase of the Moon's libration.

Professor Brown has compared the observations of the Moon with his tables in *M.N.*, **93**, 603, for the years 1923–1931, using the results of meridian observations as well as occultations. The following table shows the differences between the Cape occultations and the adopted correction:—

	Cape	Adopted	A – C		Cape	Adopted	A – C
	″	″	″		″	″	″
1923	+7·47	+7·18	−0·29	1928	+6·35	+5·54	−0·81
24	7·67	7·14	−0·53	29	5·04	5·28	+0·24
25	7·07	6·66	−0·41	30	5·59	5·01	−0·58
26	7·65	6·44	−1·21	31	5·38	4·65	−0·73
27	6·91	6·07	−0·84				

From these we may assume that the Cape observations require a correction of −0″·50 to reduce them to the adopted means for the years 1923–1931. Applying this correction, we obtain the following table, which is a continuation of the table given by Brown (*M.N.*, **93**, 614). δT is the reduction (tabulated in *A.J.*, 799) of the tabular mean longitude to that given by Fotheringham, including the change in the great empirical term; G.E.T. is the new empirical term and O – Th is the departure from gravitational theory.

	$\delta\lambda$	δT	−G.E.T.	O – Th	$10''{\cdot}62 + ''{\cdot}44t$	Res.
	″	″	″	″	″	″
1923	+7·18	−4·42	−13·53	−10·77	+10·62	−·15
24	7·14	4·57	13·49	10·92	11·06	+·14
25	6·66	4·73	13·45	11·52	11·50	−·02
26	6·44	4·89	13·39	11·84	11·94	+·10
27	6·07	5·06	13·33	12·32	12·38	+·06
28	5·54	5·22	13·26	12·94	12·82	−·12
29	5·28	5·39	13·18	13·29	13·26	−·03
30	5·01	5·56	13·10	13·65	13·70	+·05
31	4·65	5·74	13·01	14·10	14·14	+·04
32	4·55	5·92	12·91	14·28	14·58	+·30
33	3·85	6·10	12·80	15·05	15·02	−·03
34	3·28	6·28	12·69	15·69	15·46	−·23
35	3·17	6·47	12·57	15·87	15·90	+·03

The correction to Brown's tables is decreasing about 0″·35 a year. The application of the corrections determined by Fotheringham would alter the sign of the error of the tables at the present time and leave it practically unchanged in magnitude, but the error of the modified tables would then increase about 0″·55 annually instead of decreasing about 0″·35. The applica-

tion of the great empirical term has reduced the size of the residuals, but slightly increased the rate of the change relative to Brown's theory. The extrapolated values of the corrections to Brown's tables are :

	″
1936·0	+3·00
37·0	+2·65
38·0	+2·30
39·0	+1·95
40·0	+1·60

But if the occultations instead of the Greenwich limb observations are used as a standard, these values should be increased by about 0″·5.

As the constants of his tables were mainly based on the Greenwich meridian observations of the Moon's limb, Brown reduced all the observations so as to agree in the mean with the Greenwich limb observations. For the nine years 1923–1931 he found the other series of observations to require the following corrections: Washington meridian observations +0″·36, Greenwich observations of Mösting A. −0″·24, all occultations available −0″·69. The differences can be attributed to personality in the meridian observations and to the adopted mean position of Mösting A. If we adopted the occultations as a standard this would involve a correction to the meridian observations at Greenwich and Washington of +ˢ·046 and +ˢ·070 respectively. These are not exceptionally large quantities compared with personalities which have arisen in meridian observations in the past. They are associated with observations at the preceding and following limbs, and the adopted semi-diameter of the Moon when the observations at the limbs are unequal in number. It would be interesting to compare, with Brown's theory, the results derived from meridian observations and from occultations before the introduction of the self-registering micrometer to find out whether a change took place in the observations of the Moon similar to that which took place in observations of the Sun, but the number of observations available for this purpose is not very large.

A final point may be noted in connection with the accuracy of the Moon's position obtained from occultations. It is frequently stated that an error in the time of 1ˢ corresponds to about 0″·55 in the Moon's longitude. This would be true for observations made from the Earth's centre, but may be considerably in error in connection with observations from the Earth's surface. The Moon's parallax in R.A. is $\pi \cos \phi' \sin h$. If we give π its mean value of 57′, the change in the parallax in 1ˢ of time will at the Cape Observatory near the meridian amount to 0″·21. Consequently when the Moon is near the meridian an error of 1 second in the time of an occultation involves, at this Observatory, an error of only 0″·34 in the Moon's longitude.

[TABLE

No.	Date	Star	Mag.	Phase	Instrument	Obs.	U.T. h m s
	1932						
1	Jan. 1	W.Z.C. 816	7·1	R.D.	6-inch	H.H.	1 40 30·70
2	,, 5	σ Scorpii	3·0	R.D.	6 ,,	H.H.	2 7 12·08
3	,, 18	14 H Tauri	5·3	D.D.	6 ,,	H.H.	19 8 25·12
4	,, 18	,,	,,	,,	10 ,,	C.O.	25·02
5	,, 21	28 Geminorum	5·5	D.D.	6 ,,	H.H.	19 23 18·26
6	Feb. 17	211 B Aurigæ	6·3	D.D.	6 ,,	H.H.	21 29 42·30
7	,, 17	,,	,,	,,	10 ,,	C.O.	42·00
8	,, 18	53 Geminorum	5·9	D.D.	10 ,,	C.O.	18 10 30·58
9	Apr. 16	37 Leonis	5·5	D.D.	10 ,,	H.H.	19 46 59·65
10	,, 27	W.Z.C. 1404	6·7	R.D.	6 ,,	H.H.	23 32 13·29
11	May 17	α Virginis	1·2	D.D.	6 ,,	H.H.	19 22 10·77
12	,, 17	,,	,,	R.B.	6 ,,	H.H.	20 36 59·19
13	Sept. 3	50 Virginis	6·2	D.D.	6 ,,	A.W.	17 0 17·85
14	,, 6	48 B Scorpii	4·9	D.D.	6 ,,	A.W.	16 59 28·80
15	,, 7	Co. D. -27° 11527	6·6	D.D.	6 ,,	A.W.	21 14 22·47
16	Oct. 5	10 G Sagittarii	5·7	D.D.	6 ,,	A.W.	19 10 3·19
17	,, 5	Cor. B. 11489	9·0	D.D.	6 ,,	A.W.	19 28 29·07
18	,, 5	Cor. B. 11490	8·8	D.D.	6 ,,	A.W.	19 30 21·46
19	,, 11	316 B Aquarii	6·5	D.D.	6 ,,	A.W.	19 41 49·04
20	Nov. 3	W.Z.C. 1311	6·7	D.D.	6 ,,	A.W.	19 12 37·44
	1933						
21	Jan. 4	101 Piscium	6·2	D.D.	6 ,,	H.H.	19 40 2·32
22	Feb. 16	47 G Libræ	6·1	R.D.	6 ,,	H.H.	22 4 18·59
23	Mar. 16	3 Scorpii	5·9	R.D.	6 ,,	H.H.	23 14 33·65
24	Apr. 7	c Leonis	5·1	D.D.	6 ,,	H.H.	23 45 28·99
25	,, 13	116 B Scorpii	6·2	R.D.	6 ,,	H.H.	20 6 21·00
26	,, 14	163 G Ophiuchi	6·3	R.D.	6 ,,	H.H.	22 24 55·96
27	,, 15	χ Sagittarii	4·4	R.D.	6 ,,	H.H.	0 8 32·81
28	,, 30	47 Geminorum	5·6	D.D.	6 ,,	H.H.	17 53 35·32
29	June 17	W.Z.C. 70	7·1	R.D.	6 ,,	H.H.	1 45 45·11
30	July 1	B.D. -11° 3469	7·4	D.D.	10 ,,	J.	20 48 13·62
31	,, 1	,,	,,	,,	6 ,,	H.H.	13·92
32	,, 10	42 Capricorni	5·1	R.D.	6 ,,	H.H.	0 15 43·31
33	,, 17	64 Arietis	5·8	R.D.	6 ,,	H.H.	3 47 24·50
34	,, 30	17 G Libræ	6·4	D.D.	6 ,,	H.H.	18 52 3·67
35	,, 30	18 G Libræ	6·1	D.D.	10 ,,	J.	19 31 12·69
36	,, 30	,,	,,	,,	6 ,,	H.H.	13·39
37	,, 31	A Scorpii	4·6	D.D.	6 ,,	H.H.	22 34 24·46
38	Aug. 2	10 G Sagittarii	5·7	D.D.	10 ,,	J.	18 0 3·14
39	,, 2	,,	,,	,,	6 ,,	H.H.	2·94
40	,, 4	ψ Sagittarii	4·9	D.D.	6 ,,	H.H.	0 35 22·58
41	,, 11	η Piscium	3·7	D.B.	6 ,,	H.H.	4 37 25·59
42	,, 12	θ Arietis	5·7	R.D.	6 ,,	H.H.	4 28 27·15
43	,, 12	W.Z.C. 137	6·8	R.D.	6 ,,	H.H.	4 35 11·24
44	,, 29	163 G Ophiuchi	6·3	D.D.	6 ,,	H.H.	21 53 30·93
45	Sept. 8	ι Arietis	5·1	R.D.	6 ,,	H.H.	3 3 7·48
46	,, 11	W.Z.C. 298	6·6	R.D.	6 ,,	H.H.	0 47 26·46
47	Oct. 25	B.D. -21° 5684	6·6	D.D.	6 ,,	H.H.	21 55 48·32
48	Nov. 21	W.Z.C. 1325	6·8	D.D.	6 ,,	H.H.	20 22 13·27
49	,, 22	19 Capricorni	5·7	D.D.	10 ,,	J.	18 57 0·59
50	,, 28	4 Arietis	5·8	D.D.	10 ,,	J.	19 30 45·34

No.	Star's Apparent R.A.	Star's Apparent Dec.	Equations of Condition
	h m s	° ′ ″	″
1	12 35 13·44	− 4 0 1·3	$-1{\cdot}65 = -{\cdot}950\Delta\alpha + {\cdot}303\Delta\delta = -{\cdot}983\Delta\lambda - {\cdot}182\Delta\beta$
2	16 17 1·85	−25 25 56·8	$+0{\cdot}11 = -{\cdot}899 + {\cdot}029 = -{\cdot}983 - {\cdot}182$
3	3 46 14·27	+25 22 44·7	$-0{\cdot}44 = +{\cdot}746 + {\cdot}574 = +{\cdot}941 + {\cdot}339$
4	...	...	... ...
5	6 40 28·86	+29 2 36·9	$+0{\cdot}71 = +{\cdot}848 + {\cdot}273 = +{\cdot}937 + {\cdot}350$
6	6 16 53·06	+29 34 36·0	$+0{\cdot}94 = +{\cdot}405 + {\cdot}887 = +{\cdot}422 + {\cdot}907$
7	...	...	... ...
8	7 11 44·38	+28 1 9·8	$+0{\cdot}04 = +{\cdot}319 - {\cdot}933 = +{\cdot}495 - {\cdot}869$
9	10 13 3·78	+14 4 0·9	$-0{\cdot}96 = +{\cdot}754 - {\cdot}631 = +{\cdot}972 - {\cdot}235$
10	21 2 50·54	−20 27 12·4	$+0{\cdot}55 = -{\cdot}731 - {\cdot}623 = -{\cdot}949 - {\cdot}316$
11	13 21 38·62	−10 48 42·0	$-3{\cdot}11 = +{\cdot}725 - {\cdot}674 = +{\cdot}962 - {\cdot}274$
12	,,	,,	$+0{\cdot}43 = -{\cdot}784 + {\cdot}600 = -{\cdot}983 + {\cdot}183$
13	13 6 12·91	− 9 58 13·1	$+2{\cdot}28 = +{\cdot}878 - {\cdot}450 = +1{\cdot}000 + {\cdot}002$
14	15 59 16·15	−25 40 54·4	$+0{\cdot}84 = +{\cdot}882 - {\cdot}199 = +1{\cdot}000 + {\cdot}024$
15	17 9 49·13	−27 43 17·9	$-1{\cdot}06 = +{\cdot}254 + {\cdot}957 = +{\cdot}214 + {\cdot}977$
16	17 52 26·95	−28 3 31·0	$+1{\cdot}22 = +{\cdot}183 + {\cdot}978 = +{\cdot}218 + {\cdot}976$
17	17 53 53·83	−28 9 31·6	$-1{\cdot}00 = +{\cdot}753 + {\cdot}514 = +{\cdot}864 + {\cdot}504$
18	17 53 54·75	−28 8 18·4	$-0{\cdot}78 = +{\cdot}716 + {\cdot}579 = +{\cdot}823 + {\cdot}569$
19	23 16 48·11	− 4 17 1·5	$+0{\cdot}66 = +{\cdot}995 + {\cdot}056 = +{\cdot}906 - {\cdot}422$
20	19 38 18·16	−25 1 12·6	$+0{\cdot}22 = +{\cdot}333 + {\cdot}930 = +{\cdot}566 + {\cdot}824$
21	1 32 11·88	+14 19 19·2	$+0{\cdot}37 = +{\cdot}952 - {\cdot}198 = +{\cdot}805 - {\cdot}593$
22	15 2 36·05	−21 46 29·4	$-1{\cdot}69 = -{\cdot}925 - {\cdot}026 = -{\cdot}942 - {\cdot}335$
23	15 50 39·54	−25 2 58·8	$+3{\cdot}24 = -{\cdot}902 + {\cdot}058 = -{\cdot}985 - {\cdot}170$
24	10 57 18·83	+ 6 27 31·0	$+0{\cdot}70 = +{\cdot}965 - {\cdot}242 = +{\cdot}972 + {\cdot}237$
25	16 27 18·39	−26 23 48·2	$-0{\cdot}63 = -{\cdot}889 - {\cdot}088 = -{\cdot}970 - {\cdot}245$
26	17 39 6·81	−27 51 21·0	$-0{\cdot}07 = -{\cdot}878 + {\cdot}090 = -{\cdot}997 + {\cdot}078$
27	17 43 22·80	−27 48 33·3	$-0{\cdot}05 = -{\cdot}541 + {\cdot}789 = -{\cdot}616 + {\cdot}788$
28	7 7 15·03	+26 58 13·0	$+1{\cdot}91 = +{\cdot}651 - {\cdot}686 = +{\cdot}835 - {\cdot}550$
29	1 8 49·05	+11 55 46·9	$-0{\cdot}62 = -{\cdot}715 + {\cdot}685 = -{\cdot}357 + {\cdot}934$
30	13 12 24·05	−12 0 4·3	$-0{\cdot}38 = +{\cdot}659 - {\cdot}739 = +{\cdot}926 - {\cdot}378$
31	...	...	... ...
32	21 37 57·92	−14 20 37·5	$+3{\cdot}11 = -{\cdot}801 + {\cdot}559 = -{\cdot}529 + {\cdot}849$
33	3 20 22·52	+24 29 27·7	$+1{\cdot}20 = -{\cdot}778 + {\cdot}528 = -{\cdot}678 + {\cdot}735$
34	14 42 25·20	−20 53 54·0	$+0{\cdot}11 = +{\cdot}932 - {\cdot}017 = +{\cdot}952 + {\cdot}308$
35	14 43 27·38	−21 3 0·8	$-0{\cdot}56 = +{\cdot}910 - {\cdot}216 = +{\cdot}994 + {\cdot}109$
36	...	...	... ...
37	15 49 38·27	−25 8 0·1	$-0{\cdot}66 = +{\cdot}888 + {\cdot}175 = +{\cdot}927 + {\cdot}376$
38	17 52 32·07	−28 3 29·9	$+0{\cdot}68 = +{\cdot}653 - {\cdot}674 = +{\cdot}721 - {\cdot}693$
39	...	...	... ...
40	19 11 30·32	−25 22 26·0	$-0{\cdot}50 = +{\cdot}333 + {\cdot}929 = +{\cdot}543 + {\cdot}840$
41	1 27 56·61	+15 0 22·8	$-0{\cdot}65 = +{\cdot}351 + {\cdot}932 = +{\cdot}716 + {\cdot}698$
42	2 14 26·44	+19 35 49·2	$+1{\cdot}12 = -{\cdot}943 - {\cdot}079 = -{\cdot}958 + {\cdot}286$
43	2 15 27·78	+19 23 11·6	$+1{\cdot}76 = -{\cdot}162 - {\cdot}985 = -{\cdot}515 - {\cdot}857$
44	17 39 8·24	−27 51 24·0	$-0{\cdot}40 = +{\cdot}851 - {\cdot}264 = +{\cdot}962 - {\cdot}273$
45	1 53 44·81	+17 29 50·8	$+3{\cdot}05 = -{\cdot}817 - {\cdot}521 = -{\cdot}989 - {\cdot}150$
46	4 25 14·71	+27 15 40·5	$+1{\cdot}12 = -{\cdot}104 - {\cdot}993 = -{\cdot}263 - {\cdot}965$
47	20 15 34·90	−21 9 38·0	$-2{\cdot}62 = +{\cdot}567 + {\cdot}792 = +{\cdot}822 + {\cdot}570$
48	19 55 39·17	−22 23 38·3	$-0{\cdot}61 = +{\cdot}886 + {\cdot}272 = +1{\cdot}000 - {\cdot}001$
49	20 51 4·10	−18 10 33·8	$-1{\cdot}78 = +{\cdot}736 + {\cdot}630 = +{\cdot}950 + {\cdot}314$
50	1 44 37·09	+16 37 51·8	$-1{\cdot}19 = +{\cdot}334 + {\cdot}938 = +{\cdot}686 + {\cdot}728$

No.	Date	Star	Mag.	Phase	Instrument	Obs.	U.T. h m s
	1934						
51	Jan. 2	κ Geminorum	3·6	R.D.	6 inch	H.H.	0 11 54·16
52	,, 11	B.D. −20° 4123	7·3	R.D.	6 ,,	H.H.	0 10 13·79
53	Feb. 2	79 Leonis	5·5	R.D.	6 ,,	H.H.	23 41 1·34
54	,, 4	ψ Virginis	4·9	R.D.	6 ,,	H.H.	21 57 19·05
55	,, 9	W.Z.C. 1076	6·9	R.D.	6 ,,	H.H.	2 3 6·47
56	,, 19	ν Arietis	5·3	D.D.	10 ,,	J.	17 44 5·27
57	,, 21	W.Z.C. 298	6·6	D.D.	10 ,,	J.	20 41 54·10
58	,, 26	9 Cancri	6·2	D.D.	6 ,,	H.H.	0 13 45·51
59	Apr. 2	43 B Libræ	5·7	R.D.	6 ,,	H.H.	18 56 0·92
60	,, 6	66 B Sagittarii	4·7	R.D.	6 ,,	H.H.	1 13 50·80
61	,, 7	W.Z.C. 1291	7·1	R.D.	6 ,,	H.H.	4 9 11·14
62	,, 8	B.D. −21° 5684	6·6	R.D.	6 ,,	H.H.	0 57 13·29
63	May 9	λ Piscium	4·6	R.D.	6 ,,	H.H.	4 35 53·09
64	,, 26	231 G Virginis	6·4	D.D.	6 ,,	H.H.	19 53 21·54
65	June 3	W.Z.C. 1455	7·0	R.D.	6 ,,	H.H.	4 19 10·68
66	,, 20	78 B Virginis	6·5	D.D.	6 ,,	H.H.	19 34 38·49
67	,, 23	17 G Libræ	6·4	D.D.	6 ,,	H.H.	18 11 4·18
68	,, 23	18 G Libræ	6·1	D.D.	6 ,,	H.H.	18 53 32·46
69	July 15	W.Z.C. 696	7·4	D.D.	6 ,,	A.W.	17 56 0·68
70	,, 19	75 Virginis	5·6	D.D.	6 ,,	A.W.	19 47 41·54
71	,, 20	B.D. −19° 3879	7·3	D.D.	6 ,,	A.W.	19 19 43·95
72	,, 20	B.D. −19° 3880	7·6	D.D.	6 ,,	A.W.	19 21 17·00
73	,, 22	α Scorpii	1·2	D.D.	6 ,,	A.W.	21 54 20·21
74	,, 22	,,	,,	R.B.	6 ,,	A.W.	23 0 59·93
75	,, 22	116 B Scorpii	6·2	D.D.	6 ,,	A.W.	23 3 5·79
76	Aug. 2	W.Z.C. 134	7·1	R.D.	6 ,,	A.W.	1 6 29·40
77	,, 2	θ Arietis	5·7	R.D.	6 ,,	A.W.	3 35 53·34
78	,, 20	36 Ophiuchi	5·4	D.D.	6 ,,	A.W.	0 13 55·81
79	,, 21	208 B Sagittarii	6·1	D.D.	6 ,,	A.W.	19 40 42·60
80	,, 23	29 Capricorni	5·5	D.D.	10 ,,	J.	17 44 24·26
81	,, 23	,,	,,	,,	6 ,,	A.W.	24·46
82	Sept. 2	107 B Aurigæ	6·5	R.D.	6 ,,	A.W.	2 23 38·71
83	,, 12	B.D. −16° 3760	6·8	D.D.	6 ,,	A.W.	17 34 54·72
84	,, 13	17 G Libræ	6·4	D.D.	6 ,,	A.W.	17 1 15·93
85	,, 13	18 G Libræ	6·1	D.D.	6 ,,	A.W.	17 37 11·78
86	,, 16	4 G Sagittarii	6·2	D.D.	6 ,,	A.W.	19 10 40·07
87	,, 17	162 B Sagittarii	6·4	D.D.	6 ,,	A.W.	22 33 28·66
88	,, 27	64 Arietis	5·8	R.D.	6 ,,	A.W.	3 11 18·29
89	Oct. 13	151 G Ophiuchi	6·0	D.D.	6 ,,	A.W.	18 38 50·30
90	,, 18	170 B Aquarii	6·0	D.D.	6 ,,	A.W.	17 55 24·39
91	,, 24	14 H Tauri	5·3	R.D.	6 ,,	A.W.	22 30 45·97
92	Nov. 14	W.Z.C. 1476	7·0	D.D.	10 ,,	J.	18 46 28·38
93	Dec. 16	θ Arietis	5·7	D.D.	6 ,,	A.W.	22 47 22·26
	1935						
94	Jan. 10	36 Piscium	6·2	D.D.	10 ,,	J.	20 21 28·65
95	,, 10	B.D. +7° 32	7·4	D.D.	6 ,,	A.W.	20 21 36·53
96	,, 14	14 H Tauri	5·4	D.D.	10 ,,	J.	20 38 27·70
97	,, 14	,,	,,	,,	6 ,,	A.W.	27·50
98	,, 24	359 B Leonis	6·3	R.D.	6 ,,	A.W.	2 3 6·14
99	,, 26	W.Z.C. 825	6·6	R.D.	6 ,,	A.W.	1 41 34·77
100	,, 30	σ Scorpii	3·1	D.B.	6 ,,	A.W.	1 0 53·85
101	,, 30	,,	,,	R.D.	6 ,,	A.W.	1 19 23·11
102	,, 31	136 G Ophiuchi	6·3	R.D.	6 ,,	A.W.	2 33 13·73

No.	Star's Apparent R.A.	Star's Apparent Dec.	Equations of Condition
	h m s	° ′ ″	″
51	7 40 30·34	+24 33 28·3	$-0\cdot05 = -\cdot883\Delta\alpha + \cdot262\Delta\delta = -1\cdot000\Delta\lambda + \cdot012\Delta\beta$
52	14 53 35·20	−21 7 2·8	$+2\cdot40 = -\cdot588 + \cdot773 = -\cdot836 + \cdot549$
53	11 20 41·19	+ 1 46 1·9	$+0\cdot94 = -\cdot979 - \cdot205 = -\cdot772 - \cdot636$
54	12 50 56·68	− 9 11 2·1	$-0\cdot51 = -\cdot151 + \cdot988 = -\cdot570 + \cdot821$
55	16 39 47·34	−26 41 9·8	$-0\cdot25 = -\cdot878 + \cdot167 = -\cdot998 + \cdot062$
56	2 35 4·57	+21 40 47·7	$-1\cdot86 = +\cdot723 + \cdot633 = +\cdot936 + \cdot353$
57	4 25 16·43	+27 15 47·5	$-1\cdot90 = +\cdot703 - \cdot617 = +\cdot703 - \cdot711$
58	8 2 26·49	+22 49 30·2	$-3\cdot13 = +\cdot837 - \cdot426 = +\cdot989 - \cdot147$
59	14 53 39·14	−21 7 27·4	$-0\cdot47 = -\cdot833 + \cdot442 = -\cdot988 + \cdot158$
60	18 13 57·44	−27 4 9·4	$+0\cdot43 = -\cdot271 - \cdot953 = -\cdot382 - \cdot924$
61	19 21 23·62	−24 32 37·0	$+2\cdot14 = -\cdot863 - \cdot314 = -\cdot995 - \cdot102$
62	20 15 36·28	−21 9 31·9	$+1\cdot84 = -\cdot768 - \cdot565 = -\cdot957 - \cdot291$
63	23 38 41·64	+ 1 25 5·3	$+0\cdot30 = -\cdot543 - \cdot840 = -\cdot866 - \cdot499$
64	14 13 27·73	−18 17 3·8	$+1\cdot11 = +\cdot939 + \cdot136 = +\cdot883 + \cdot468$
65	21 43 11·35	−11 59 55·1	$+0\cdot31 = -\cdot948 + \cdot239 = -\cdot787 + \cdot617$
66	12 10 54·83	− 5 21 22·7	$+0\cdot85 = +\cdot986 + \cdot134 = +\cdot827 + \cdot563$
67	14 42 29·27	−20 54 10·3	$+1\cdot11 = +\cdot866 - \cdot374 = +\cdot997 - \cdot078$
68	14 43 31·45	−21 3 16·9	$-2\cdot53 = +\cdot842 - \cdot431 = +\cdot990 - \cdot142$
69	10 27 2·01	+ 7 23 44·2	$-2\cdot97 = +\cdot732 - \cdot675 = +\cdot960 - \cdot279$
70	13 29 22·34	−15 1 45·2	$-1\cdot50 = -\cdot211 - \cdot976 = +\cdot177 - \cdot984$
71	14 21 49·14	−19 40 24·9	$-2\cdot55 = +\cdot728 - \cdot633 = +\cdot938 - \cdot347$
72	14 21 51·32	−19 40 40·0	$-3\cdot20 = +\cdot727 - \cdot635 = +\cdot937 - \cdot349$
73	16 25 25·29	−26 17 28·6	$-0\cdot60 = +\cdot831 - \cdot372 = +\cdot965 - \cdot264$
74	,,	,,	$+2\cdot06 = -\cdot827 - \cdot382 = -\cdot877 - \cdot480$
75	16 27 23·57	−26 23 59·0	$+1\cdot31 = +\cdot560 - \cdot780 = +\cdot707 - \cdot707$
76	2 10 14·56	+19 18 38·2	$+1\cdot41 = -\cdot947 + \cdot016 = -\cdot933 + \cdot359$
77	2 14 29·77	+19 36 4·5	$+1\cdot36 = -\cdot315 - \cdot943 = -\cdot631 - \cdot775$
78	17 11 21·04	−26 30 40·7	$-2\cdot76 = +\cdot825 + \cdot376 = +\cdot918 + \cdot396$
79	19 11 36·52	−24 17 34·0	$-1\cdot99 = +\cdot658 + \cdot690 = +\cdot849 + \cdot528$
80	21 12 10·27	−15 26 33·0	$-1\cdot27 = +\cdot780 + \cdot584 = +\cdot972 + \cdot235$
81	...	...	... ...
82	5 31 50·09	+27 37 20·5	$+0\cdot65 = -\cdot698 - \cdot621 = -\cdot777 - \cdot630$
83	13 50 47·94	−16 51 37·6	$-0\cdot84 = +\cdot652 - \cdot732 = +\cdot901 - \cdot435$
84	14 42 28·33	−20 54 6·6	$-1\cdot28 = +\cdot902 + \cdot250 = +\cdot852 + \cdot523$
85	14 43 30·51	−21 3 13·3	$-2\cdot28 = +\cdot928 - \cdot087 = +\cdot978 + \cdot207$
86	17 44 23·86	−26 57 21·0	$+0\cdot57 = +\cdot447 - \cdot866 = +\cdot464 - \cdot886$
87	18 54 21·84	−24 58 1·3	$+2\cdot50 = +\cdot505 - \cdot830 = +\cdot401 - \cdot916$
88	3 20 28·71	+24 29 49·1	$+0\cdot41 = -\cdot878 + \cdot287 = -\cdot867 + \cdot499$
89	17 27 41·62	−26 13 23·6	$+0\cdot37 = +\cdot148 + \cdot986 = +\cdot179 + \cdot984$
90	22 20 8·77	− 7 31 20·4	$+0\cdot55 = +\cdot702 + \cdot705 = +\cdot944 + \cdot329$
91	3 46 25·64	+25 23 13·8	$+0\cdot82 = -\cdot906 + \cdot090 = -\cdot962 + \cdot274$
92	22 6 1·37	− 8 30 20·5	$-2\cdot69 = +\cdot212 + \cdot977 = +\cdot609 + \cdot794$
93	2 14 31·87	+19 36 17·8	$-0\cdot46 = +\cdot718 - \cdot649 = +\cdot511 - \cdot859$
94	0 13 14·14	+ 7 52 53·2	$+0\cdot20 = -\cdot313 + \cdot949 = +\cdot126 + \cdot992$
95	0 14 32·73	+ 7 30 25·2	$-1\cdot86 = +\cdot886 - \cdot450 = +\cdot610 - \cdot793$
96	3 46 26·23	+25 23 17·1	$+0\cdot42 = +\cdot882 - \cdot242 = +\cdot913 - \cdot409$
97	...	...	... ...
98	11 20 0·31	+ 0 29 9·6	$+1\cdot06 = -\cdot719 + \cdot694 = -\cdot954 + \cdot300$
99	12 48 1·45	− 9 59 15·6	$+0\cdot33 = -\cdot403 + \cdot912 = -\cdot751 + \cdot660$
100	16 17 14·65	−25 26 22·6	$+0\cdot97 = +\cdot032 - \cdot999 = +\cdot162 - \cdot987$
101	,,	,,	$+1\cdot78 = -\cdot522 - \cdot816 = -\cdot472 - \cdot881$
102	17 22 54·32	−25 53 18·5	$+1\cdot18 = -\cdot073 + \cdot997 = -\cdot081 + \cdot997$

No.	Date	Star	Mag.	Phase	Instrument	Obs.	U.T. h m s
	1935						
103	Feb. 1	Co. D. −25° 13170	6·2	R.D.	6 inch	A. W.	2 15 0·96
104	,, 24	43 B Libræ	5·8	R.D.	6 ,,	A. W.	21 13 33·49
105	,, 25	47 G Libræ	6·1	R.D.	6 ,,	A. W.	1 38 6·71
106	,, 25	40 B Scorpii	5·4	R.D.	6 ,,	A. W.	23 45 8·32
107	,, 26	50 B Scorpii	6·4	R.D.	6 ,,	A. W.	2 23 46·15
108	,, 26	W.Z.C. 1087	7·1	R.D.	6 ,,	A. W.	22 47 0·00
109	Mar. 8	W.Z.C. 134	7·1	D.D.	6 ,,	A. W.	17 49 12·02
110	,, 24	169 B Libræ	5·8	R.D.	6 ,,	A. W.	20 15 4·85
111	Apr. 12	81 Cancri	6·4	D.D.	10 ,,	J.	21 12 6·39
112	,, 15	359 B Leonis	6·3	D.D.	10 ,,	J.	20 7 36·47
113	,, 15	,,	,,	,,	Heliom.	A. W.	36·27
114	,, 23	136 G Ophiuchi	6·3	R.D.	,,	A. W.	0 4 27·53
115	,, 23	70 B Sagittarii	6·4	R.D.	,,	A. W.	21 16 48·27
116	,, 26	π Capricorni	5·2	R.D.	,,	A. W.	1 32 4·50
117	,, 26	W.Z.C. 1361	6·7	R.D.	,,	A. W.	2 22 4·68
118	,, 27	18 Aquarii	5·5	R.D.	,,	A. W.	2 20 54·26
119	May 12	p^4 Leonis	5·7	D.D.	,,	A. W.	19 26 21·27
120	,, 21	191 B Sagittarii	6·5	R.D.	,,	A. W.	22 20 22·01
121	,, 22	222 B Sagittarii	5·6	R.D.	,,	A. W.	5 16 11·37
122	June 5	θ Cancri	5·6	D.D.	,,	A. W.	16 40 16·35
123	,, 14	42 Libræ	5·1	D.D.	,,	A. W.	19 27 48·82
124	,, 26	θ Arietis	5·7	R.D.	,,	A. W.	5 22 42·15
125	,, 26	W.Z.C. 137	6·8	R.D.	,,	A. W.	5 31 15·34
126	July 23	ι Arietis	5·2	R.D.	,,	A. W.	0 48 6·76
127	Aug. 4	W.Z.C. 805	6·7	D.D.	,,	A. W.	17 14 20·77
128	,, 7	43 B Libræ	5·8	D.D.	,,	A. W.	20 38 55·97
129	,, 11	154 B Sagittarii	5·9	D.D.	,,	A. W.	22 9 25·88
130	,, 12	168 B Sagittarii	6·3	D.D.	,,	A. W.	0 53 0·93
131	Sept. 10	47 B Capricorni	6·2	D.D.	,,	A. W.	0 27 39·67
132	Oct. 1	64 G Libræ	5·7	D.D.	,,	A. W.	17 46 55·43
133	,, 2	W.Z.C. 1035	6·6	D.D.	,,	A. W.	17 47 30·18
134	,, 2	41 G Scorpii	6·3	D.D.	,,	A. W.	19 24 28·93
135	,, 2	W.Z.C. 1047	6·6	D.D.	,,	A. W.	20 35 30·85
136	,, 3	W.Z.C. 1104	7·2	D.D.	,,	A. W.	18 33 37·84
137	,, 6	W.Z.C. 1335	7·0	D.D.	,,	A. W.	21 50 15·61
138	,, 7	95 B Capricorni	6·0	D.D.	,,	A. W.	18 18 58·30
139	,, 14	17 Tauri	3·8	D.B.	,,	A. W.	20 56 23·92
140	,, 14	q Tauri	4·4	D.B.	,,	A. W.	21 20 10·22
141	,, 14	20 Tauri	4·0	D.B.	,,	A. W.	21 26 14·72
142	,, 14	17 Tauri	3·8	R.D.	,,	A. W.	21 52 6·76
143	,, 14	16 Tauri	5·4	R.D.	,,	A. W.	22 3 43·45
144	,, 14	q Tauri	4·4	R.D.	,,	A. W.	22 16 56·08
145	,, 14	20 Tauri	4·0	R.D.	,,	A. W.	22 36 51·50
146	,, 14	21 Tauri	5·8	R.D.	,,	A. W.	22 38 23·85
147	,, 14	22 Tauri	6·5	R.D.	,,	A. W.	22 45 3·95
148	Nov. 1	ν^1 Sagittarii	5·0	D.D.	,,	A. W.	20 39 46·99
149	,, 1	ν^2 Sagittarii	5·0	D.D.	,,	A. W.	21 3 43·36
150	Dec. 2	44 Aquarii	5·8	D.D.	6-inch	J.	20 10 7·59
151	,, 2	,,	,,	,,	Heliom.	A. W.	7·49
152	,, 6	4 Arietis	5·7	D.D.	,,	A. W.	18 18 15·89
153	,, 8	η Tauri	3·0	D.D.	10-inch	J.	18 43 8·05
154	,, 8	,,	,,	,,	6 ,,	S.	8·30
155	,, 8	,,	,,	,,	Heliom.	A. W.	7·15

No.	Star's Apparent R.A. h m s	Dec. ° ′ ″	Equations of Condition ″
103	18 25 21·66	−25 18 3·4	+1·75 = −·752Δα − ·552Δδ = − ·895Δλ − ·446Δβ
104	14 53 41·95	−21 7 38·3	+0·85 = −·922 + ·129 = − ·990 −·141
105	15 2 43·88	−21 46 58·2	+0·36 = −·063 + ·998 = − ·318 +·948
106	15 54 42·38	−24 38 49·9	+0·86 = −·404 − ·895 = − ·290 −·957
107	16 0 2·09	−24 33 3·5	−2·88 = +·011 +1·000 = − ·143 +·990
108	16 50 22·51	−25 43 30·5	+2·18 = −·896 − ·054 = − ·993 −·115
109	2 10 15·65	+19 18 47·0	−1·46 = +·926 − ·208 = + ·858 −·514
110	15 34 1·16	−22 55 49·5	+1·42 = −·110 + ·993 = − ·319 +·948
111	9 8 46·80	+15 15 25·0	−0·23 = +·922 − ·302 = + ·997 +·074
112	11 20 1·06	+ 0 29 3·0	−0·51 = +·946 − ·325 = + ·992 +·124
113	...	...	... ...
114	17 22 57·01	−25 53 20·5	−0·28 = −·864 − ·277 = − ·963 −·270
115	18 17 34·26	−24 56 49·0	+0·28 = −·840 − ·369 = − ·964 −·267
116	20 23 38·28	−18 25 30·1	−0·51 = −·936 − ·152 = − ·986 +·167
117	20 25 20·26	−18 5 21·4	−0·89 = −·718 + ·653 = − ·511 +·860
118	21 20 40·21	−13 9 24·9	+1·30 = −·952 + ·200 = − ·830 +·557
119	11 3 37·84	+ 2 18 16·1	−0·30 = +·964 + ·266 = + ·751 +·660
120	19 4 52·39	−23 17 39·1	+2·68 = −·743 − ·586 = − ·909 −·417
121	19 16 47·90	−22 31 27·4	+2·35 = −·563 − ·792 = − ·772 −·635
122	8 27 55·19	+18 18 51·4	+0·16 = +·483 − ·862 = + ·760 −·649
123	15 36 29·95	−23 36 46·2	−3·42 = +·748 − ·578 = + ·910 −·415
124	2 14 32·14	+19 36 14·8	+1·53 = −·944 − ·065 = − ·969 +·247
125	2 15 33·48	+19 23 37·1	+2·01 = −·264 − ·960 = − ·562 −·827
126	1 53 50·55	+17 30 18·2	+1·45 = −·678 − ·705 = − ·905 −·424
127	12 24 37·23	− 8 19 19·9	+0·71 = −·329 − ·943 = + ·088 −·996
128	14 53 43·20	−21 7 46·2	−3·55 = +·437 + ·882 = + ·237 +·972
129	18 52 9·20	−23 15 27·5	+0·53 = −·096 + ·995 = + ·078 +·997
130	18 57 47·39	−22 47 15·0	−0·63 = +·140 + ·988 = + ·341 +·940
131	20 31 56·24	−16 44 47·7	−1·61 = +·867 + ·419 = + ·995 +·096
132	15 12 39·24	−22 9 49·3	−3·99 = +·705 − ·647 = + ·884 −·468
133	16 6 18·08	−24 24 56·4	−1·03 = +·495 − ·839 = + ·643 −·766
134	16 9 53·71	−24 15 38·5	−0·57 = +·868 − ·297 = + ·983 −·184
135	16 12 34·85	−24 7 28·3	−1·46 = +·899 + ·145 = + ·967 +·255
136	17 4 0·30	−24 54 57·5	−1·74 = +·436 + ·876 = + ·468 +·884
137	20 4 32·02	−18 59 26·2	+0·16 = +·760 − ·594 = + ·596 −·803
138	20 55 10·43	−14 43 50·4	−0·36 = +·311 + ·947 = + ·634 +·773
139	3 41 5·59	+23 54 55·4	+1·25 = +·894 − ·228 = + ·918 −·397
140	3 41 25·02	+24 16 11·5	+1·96 = +·516 + ·827 = + ·699 +·715
141	3 42 2·28	+24 10 15·5	−0·56 = +·842 + ·398 = + ·973 +·231
142	3 41 5·59	+23 54 55·4	+5·66 = −·538 − ·810 = − ·718 −·696
143	3 41 1·09	+24 5 28·5	+2·32 = −·881 − ·280 = − ·994 −·110
144	3 41 25·02	+24 16 11·5	+2·50 = −·888 + ·250 = − ·911 +·412
145	3 42 2·28	+24 10 15·5	+2·74 = −·874 − ·302 = − ·991 −·136
146	3 42 6·85	+24 21 28·5	+2·36 = −·851 + ·372 = − ·851 +·525
147	3 42 15·26	+24 19 53·2	+2·36 = −·896 + ·214 = − ·927 +·376
148	18 50 18·22	−22 49 34·4	+0·74 = +·761 − ·561 = + ·714 −·700
149	18 51 14·86	−22 45 13·2	+0·15 = +·765 − ·555 = + ·719 −·696
150	22 13 46·66	− 5 42 22·5	−1·93 = +·404 + ·914 = + ·747 +·665
151	...	...	... ...
152	1 44 44·08	+16 38 27·8	−0·36 = +·899 − ·353 = + ·762 −·647
153	3 43 42·74	+23 54 40·5	+0·54 = +·585 − ·771 = + ·502 −·865
154	...	...	... ...
155	...	...	... ...

Notes

6, 7, 13, 22, 24, 28, 43, 50, 53, 65, 90, 110, 127, 129, 135. Conditions poor.
34, 67, 84. 17 G Libræ. Gradual, suspected double. Duplicity subsequently confirmed.
43, 94. Time altered 1^m.
51. Near full Moon.
98, 133, 135. Uncertain.
100. Disappeared in a distinct depression in limb.
108. Gradual.
121, 152. Twilight.
139, 140, 141. Very difficult.

Star Places

Nos.	From
1, 43, 48, 65, 92, 99, 117, 125, 137.	*First Cape Catalogue*, 1925.
10, 15, 20, 29, 46, 47, 55, 57, 61, 62, 69, 72, 76, 103, 108, 109, 127, 133, 135, 136.	*Second Cape Catalogue*, 1925 (not yet printed).
17, 18, 30.	Current Cape observations.
52.	*Boss' Catalogue.*
71.	No. 72, corrected for $\Delta\alpha$ and $\Delta\delta$.
83.	*N.A.*, 1936.
95.	*Cape General Catalogue*, 1900.

17. R.A. and Dec. 1932·0, $17^h\ 53^m\ 51^s{\cdot}05$, $-28°\ 9'\ 20''{\cdot}2$.
18. ,, ,, ,, ,, , $17^h\ 53^m\ 51^s{\cdot}97$, $-28°\ 8'\ 7''{\cdot}0$.
30. ,, ,, ,, 1933·0, $13^h\ 12^m\ 21^s{\cdot}81$, $-11°\ 59'\ 46''{\cdot}4$.

REMARKS ON AN APPARENT LUNAR EFFECT IN TIME DETERMINATIONS.

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1. In their paper "Further Investigations of an Apparent Lunar Effect in Time Determinations" * A. L. Loomis and H. T. Stetson made an analysis of the day-by-day time comparisons between each two of the three stations, Rugby, Bordeaux and Annapolis, and ascertained that the longitude between Paris and Washington varies with the declination and hour-angle of the Moon. Curves representing this variation were found to take essentially the same form as those which represent the variation in longitude between Greenwich and Washington. In their results it is noticeable that the phase of this variation is inverted according as the Moon changes its declination from north to south of the equator, and attention was called to this characteristic reversal as strong evidence of a tidal effect in the region of latitude 45° on the surface of the Earth.

As to the lunar effect in latitude observations, I have already published some results of my study,† and it was those results that led me to examine

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† *Proc. Imp. Acad.*, **11**, 398, 1935.

these lunar correlations with longitude. My view is that, if the day-by-day time comparisons varied systematically over a period of a year, then the method of investigation, which was followed by Loomis and Stetson, inevitably makes one reach conclusions whereby longitudes seem as if they were really varying by the effect of the Moon, even though variations in longitude might actually have no connection with the Moon's position.

In Loomis and Stetson's paper the discrepancies in time determined at two observatories were represented by d, and the numerical values of $2d$ were derived from each day's intercomparisons of signals at the above-mentioned stations. Those values of $2d$ for each pair of stations were represented again by $x=(GBR–NSS)$, $y=(FYL–NSS)$ and $z=(FYL–GBR)$. Fortunately all values of x, y and z were given in their paper together with the necessary elements of the Moon, so that we can utilise those data for the following investigations.

2. In the first place, taking simply the monthly mean of x, y and z for each year, it was found that they all vary systematically as shown in Table I,

TABLE I

Monthly Mean Values of x, y and z
(unit $0^s.001$)

Month	1929			1930			1931		
	x	y	z	x	y	z	x	y	z
I	...	...	...	+ 81	+ 80	− 3	+38	+111	+ 52
II	+158	+128	− 52	− 8	+ 85	+ 85	−60	− 17	+ 13
III	+110	+118	+ 20	+ 50	+ 38	− 14	−62	− 10	+ 35
IV	+ 63	+103	+ 23	0	− 1	0	−24	+ 9	+ 23
V	− 50	+127	+163	− 81	− 49	+ 21	−35	+ 55	+ 85
VI	− 40	+120	+130	− 43	− 40	− 13	−58	+ 55	+103
VII	− 70	+ 60	+120	− 78	− 23	+ 58	−67	+ 37	+ 98
VIII	− 16	+ 61	+ 50	−110	− 13	+ 77	−10	− 6	+ 1
IX	− 33	+100	+115	...	...	...	+13	− 38	− 61
X	+ 6	+138	+128	−130	+ 35	+140	+18	− 1	− 32
XI	+ 39	+ 15	− 41	+ 5	+ 68	+ 53	+83	+ 35	− 68
XII	+ 7	+ 53	+ 43	− 5	+159	+153	+82	+ 45	− 51

and we see that the annual variation in x and y is particularly conspicuous. To determine the ranges of these annual terms in x and y data for the whole three years were then considered altogether, with the results shown in the second and third columns of Table II. In the first column of this table are given the mean dates for the whole period concerned, and the last column, headed by n, shows the number of data utilised. Contrary to the remarkable variations in x and y, we notice in the penultimate column of Table II that z does not change systematically in the course of a year, and this fact reminds us of Loomis and Stetson's results, where no variation with the Moon's

TABLE II

Annual Variation in x, y and z
(unit $0^s.001$)

Date	x	y	z	n
I 15	+56	+99	+30	20
II 14	+34	+83	+33	20
III 13	+31	+39	+ 5	23
IV 17	+ 4	+24	+13	19
V 17	−59	+13	+63	27
VI 12	−49	+20	+54	9
VII 19	−71	+24	+91	13
VIII 16	−45	+14	+43	21
IX 17	+ 4	− 9	−24	19
X 17	+ 1	+58	+46	25
XI 17	+62	+33	−48	39
XII 13	+35	+96	+50	25

position was found to take place in z, while striking lunar effects in x and y were ascertained.

To make the connection between the annual variation and the lunar effects clear two kinds of figures were drawn. First, monthly mean values of x and y, which are contained in the second and the third columns of Table II, were plotted, and smooth curves were then drawn through them to represent those values as closely as possible, consideration being paid to the relative weight of each value. Thus figs. 1_x and 1_y were formed. Curves in figs. 2_x and 2_y represent the lunar effects in x and y respectively. They were constructed from the numerical values given in a table on p. 464 of Loomis and Stetson's paper. In both of these two figures the full-line curve represents the variation of $2d$ with the hour-angle of the Moon, when the declination is between $+12°$ and $+28°$, the hour angle for this curve being indicated at the top of the figure. For the case where the Moon is south of the equator, from 12° S. to 28° S., the variation of $2d$ is shown by the broken-line curve. But in this case the hour-angle should be read with the scale given at the bottom of the figure.

It is easily noticed that the three curves in figs. 1_x and 2_x resemble each other strikingly. The remaining three curves in figs. 1_y and 2_y are also similar. This resemblance is a matter of course, because these three curves came from one and the same origin, as explained as follows :—

As elements of the Moon, Loomis and Stetson adopted the hour-angle and declination for the Greenwich meridian at 0^h U.T. ; but, as we know, the mean value of the right ascension of the Moon for declinations between $+12°$ and $+28°$ is 6^h throughout the whole year. On the other hand, the Greenwich sidereal time for 0^h U.T. at the beginning of a year is about $6^h.6$. This means that the hour-angle of the Moon of such declination referred to the Greenwich meridian at 0^h U.T. is about $0^h.6$ at the beginning

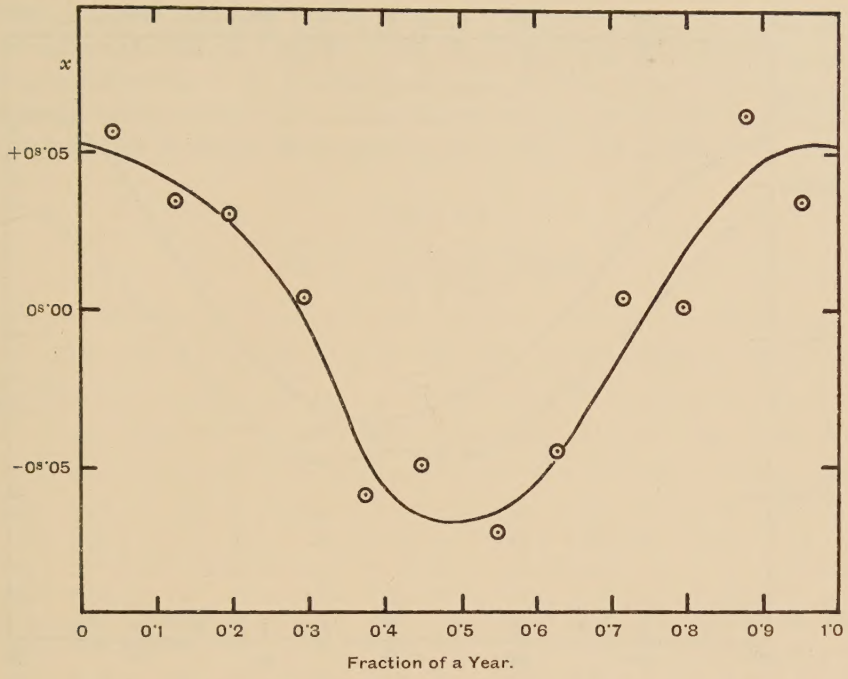

FIG. 1$_x$.—Annual Variation in x.

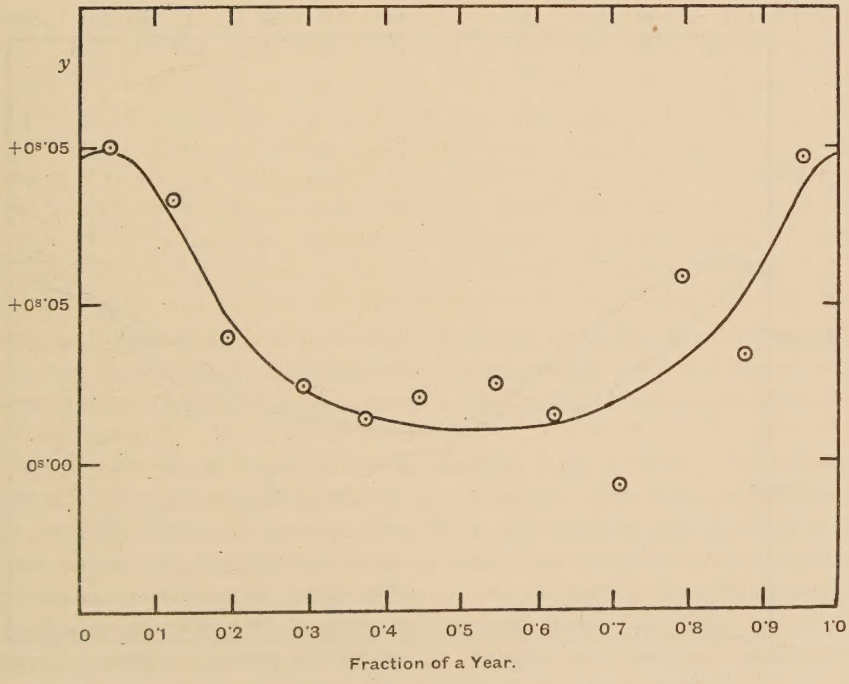

FIG. 1$_y$.—*Annual Variation in y.*

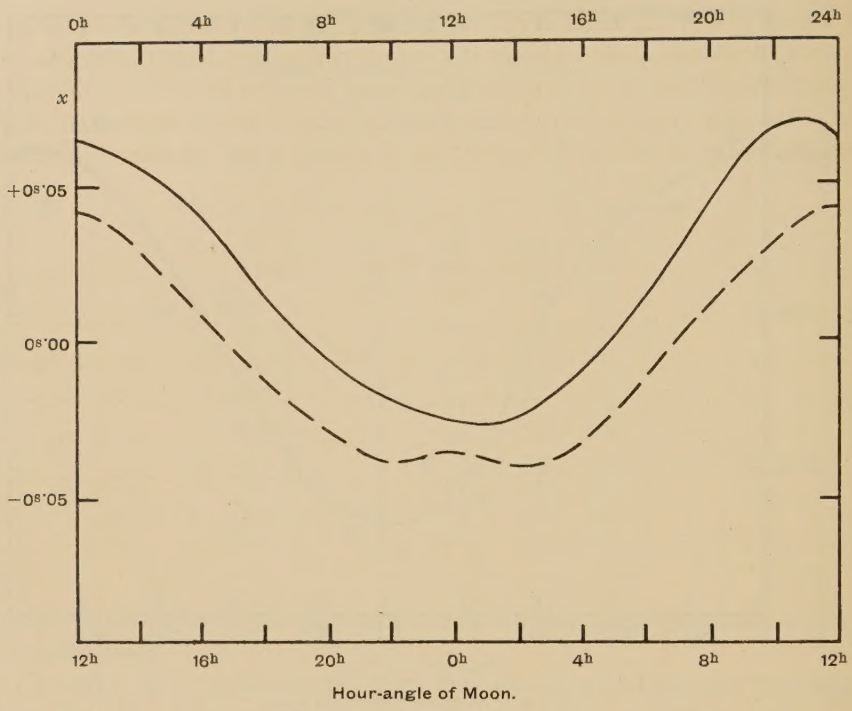

FIG. 2_x.—*Lunar effect in x.*

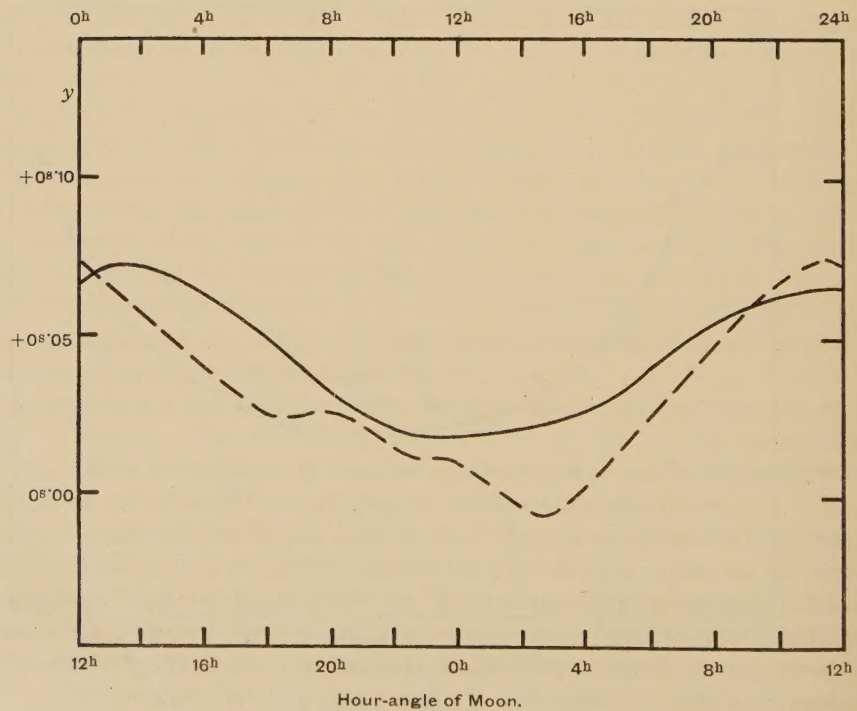

FIG. 2_y.—*Lunar effect in y.*

of a year and it advances regularly by $2^h.4$ per every tenth of a year as the sidereal time advances. Hence if the case where the Moon's declination is always greater than $+12°$ is considered alone, then the value of $2d$ corresponding to the hour-angle $0^h.6$ is nothing but the mean value of $2d$ over some short period of time of which the beginning of a year is the middle. In this way

TABLE III

Δx, Δy and Moon's Position
(unit $0^s.001$)

Hour-angle	Dec. +28° to +12°			Dec. +12° to −12°			Dec. −12° to −28°			Dec. +28° to −28°		
	Δx	Δy	n	Δx	Δy	n	Δx	Δy	n	Δx	Δy	n
h												
0·0	+43	+ 3	4	+15	0	2	−70	−50	2	+ 8	−11	8
1·2	−32	+ 3	6	− 2	+ 4	5	−18	−23	4	−18	− 3	15
2·4	+11	+ 1	10	−13	+ 2	6	0	−30	2	+ 2	− 2	18
3·6	−22	− 4	5	+13	+15	6	− 3	+ 8	4	− 3	+ 7	15
4·8	−40	−25	4	+ 3	+20	4	−20	−26	7	−19	−13	15
6·0	−10	+15	2	+10	−40	1	− 1	−27	7	− 2	−20	10
7·2	+ 4	−14	7	−18	− 5	6	+20	+30	2	− 3	− 5	15
8·4	+30	+33	4	+35	+20	2	−10	− 7	6	+11	+11	12
9·6	+ 3	+13	3	+63	+30	3	0	+ 3	4	+20	+14	10
10·8	+28	+ 5	4	+35	+18	6	−16	+11	7	+12	+12	17
12·0	+10	−20	1	+10	+20	4	− 3	−10	3	+ 5	+ 4	8
13·2	+10	+10	3	−37	− 3	3	+ 8	+22	5	− 4	+12	11
14·4	+28	+15	6	− 4	−18	5	−10	+ 8	4	+ 7	+ 2	15
15·6	+ 3	0	4	− 3	0	4	−30	−27	7	−14	−13	15
16·8	− 4	+ 8	5	−23	−13	4	−10	−28	5	−11	−11	14
18·0	+11	+ 6	7	−17	−13	3	− 2	+20	5	+ 1	+ 7	15
19·2	− 3	− 3	3	+20	+30	2	−25	0	4	− 8	+ 6	9
20·4	−12	+26	5	+15	+30	4	+60	+10	4	+18	+22	13
21·6	+38	+ 2	6	−23	−13	3	+37	0	3	+23	− 3	12
22·8	+33	+12	6	− 5	+ 3	4	+ 3	+10	3	+15	+ 8	13

the mean value of $2d$ at 0·1 of a year becomes $2d$ at the Moon's hour-angle $3^h.0$, and 0·2 of a year corresponds to the hour-angle $5^h.4$, and so on. This is the reason why the full-line curve in fig. 2 shows the same form as the curve in fig. 1.

When the Moon is south of the equator from $-12°$ to $-28°$ its right ascension should be regarded as 18^h in the mean. This value is different by 12^h from the former one, and hence it is self-evident that the broken-line curve, whose origin is shifted by 12^h to that of the full-line curve, should also take the same form as the curve in fig. 1. Accordingly it is easily understood that the phase of the lunar correlation curve should naturally be inverted according as the Moon's declination is changed from north to south of the equator, and that this inversion of phase bears no deep meaning.

Thus it is proved that, if an annual term were contained in the data, the

Moon's position seems as if it were causing some effect on those data, although such effect is of course spurious. Moreover, it was also made clear that, so far as the time comparisons for the years 1929 to 1931 are concerned, both x and y show remarkable variation over a period of a year. Nothing is known at present about the origin of this annual variation, but no one will be able to insist that such annual variation is caused by the Moon. Hence, perhaps the lunar effect derived by Loomis and Stetson is not real, and what we should inquire into is the cause of the annual variation and not the origin of lunar correlation with time comparison.

3. Smooth curves were drawn to represent the mean values of x and y, which are given in Table I. Their daily variations were then found from these curves and subtracted from each individual datum. Residuals, Δx's and Δy's, thus obtained were arranged with respect to the corresponding declination and hour-angle of the Moon, and their mean values were taken. Results are given in Table III, where n means the number of data employed. This table shows that when the variations which are not likely caused by the change in the Moon's position were rejected from the data no such results as given in Loomis and Stetson's paper could be obtained. There exists no lunar diurnal variation, nor is any reversal in phase between declination south and declination north noticeable in this table. On the contrary, the column under Dec. $+28°$ to $-28°$ suggests the existence of lunar semi-diurnal variations in x and y. But their reality is still very doubtful because of the lack of data.

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